Two-Agent New Keynesian Model with Non-Separable Preferences

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Abstract

This study employs a two-agent New Keynesian model with non-separable preferences to analyze monetary policy's impact on the economy. Unlike representative-agent New Keynesian models, the complementarity between consumption and hours worked plays a significant role in the general equilibrium elasticity of intertemporal substitution, marginal propensity to consume, and objective function for optimal monetary policy. With countercyclical inequality, procyclical savers' consumption, and a sufficiently large income effect, both the general equilibrium elasticity of intertemporal substitution and marginal propensity to consume increase with the degree of complementarity and with the ratio of hand-to-mouth households. More complementarity, more hand-to-mouth households, and a larger inequality size result in the weight of the output in the objective function being larger than that of inflation. These findings contest the separable preference paradigm in HANK literature and offer analytical frameworks for central banks to adjust reaction functions informed by household consumption-labor interactions and inequality dynamics.

Keywords: TANK, non-separable preferences, complementarity, EIS, MPC, optimal monetary policy

JEL classification: E21, E31, E40, E44, E50, E52, E58

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1 Introduction

Heterogeneous-agent New Keynesian (HANK) models have profoundly influenced contemporary macroeconomics (Kaplan and Violante, 2018). The HANK literature addresses critical macroeconomic issues, such as monetary policy transmission, credit policies the development of automatic stabilizers and fiscal policy. Economists have employed HANK models to comprehend financial crises, such as the Great Recession of 2007–2009. Much of this literature relies on numerically solved calibrated models, resulting in ambiguity regarding how the results relate to broader economic structures.

To interpret the HANK model's numerical outcomes, researchers frequently depend on the two-agent New Keynesian (TANK) model's results. Debortoli and Galí (2024) show that a well-configured TANK model effectively captures the overall impact of the HANK model even without accounting for individual income variability. Likewise, Bilbiie's analytical studies (2024, 2020b, 2008) employ TANK and THANK (i.e., tractable HANK) models to demonstrate that the ratio of hand-to-mouth households facing liquidity constraints plays a significant role in the general equilibrium elasticity of intertemporal substitution (GE-EIS), marginal propensity to consume (MPC), and an objective function for the optimal monetary policy objective function. However, Bilbiie's TANK models and most HANK models adopt separable utility functions that lack complementarity between consumption and hours worked.

Empirical evidence supports complementarity between consumption and hours. Hall (2009) posits that marginal utility increases when an individual transitions from unemployment to employment or works longer hours. Further, he reviews evidence supporting complementarity, such as the consumption retirement puzzle by Aguiar and Hurst (2005). Bilbiie's sequential studies (2020a, 2011, 2009) employ general non-separable preferences to generate positive fiscal multipliers on consumption and, thus, resolve the problem of low elasticity of intertemporal substitution combined with low income-wealth effects. However, Bilbiie (2020a) considers only representative-agent New Keynesian (RANK) models; as Bilbiie (2020a) suggests, in a non-distorted RANK model, monetary policy's impact remains unchanged despite complementarity.

A TANK economy may undergo changes in the impact of monetary policy because of complementarity. When monetary policy exerts a distinct influence, understanding its nature becomes essential. A potentially substantial effect would require a comprehensive reconstruction of the model. Moreover, in a more realistic HANK model, insights derived from the TANK model help interpret numerical results. Accordingly, I analytically examine monetary policy's impact on the economy by employing a TANK model with non-separable preferences.

I find that the complementarity between consumption and hours is crucial in three respects. First, complementarity alters the GE-EIS or the slope of the investment-savings (IS) curve in the TANK model. Second, it influences the MPC or the slope of the consumption function. Specifically, with countercyclical inequality, procyclical savers' consumption, and a sufficiently large income effect, complementarity amplifies the effect on monetary policy. Third, it crucially affects the optimal monetary policy's objective function in the TANK model. These findings challenge the separable preference paradigm in HANK literature and provide analytical foundations for central banks to recalibrate reaction functions based on household consumption-labor interactions and inequality dynamics.

Relatedly, Auclert et al. (2023)'s study incorporates complementarity in the HANK model and identifies empirical challenges that the model struggles to explain, even with a general utility function. However, they demonstrate that introducing wage rigidity can resolve these issues even with a separable utility function. Their utility functions differ from the ones used in this study. Although their conclusions rely on numerical calculations, my research is entirely analytical.

The remainder of this paper is organized as follows: Section 2 elucidates the proposed TANK model with non-separable preferences. I explain the non-separable utility function and, then, briefly outline the TANK model. Section 3 presents the main results and examines how complementarity impacts the NK-IS curve, consumption function, and objective function of the optimal policy. Section 4 concludes.

2 Model Economy

The proposed model closely resembles TANK models (Bilbiie, 2024, 2020b, 2008), except for the period utility function's shape. I first describe the utility functions and, then, provide an overview of the model as well as the set of equations derived thereof. All variables in the model include a time index in general. The absence of a time index indicates a steady state. The log deviations from steady states are denoted by lowercase letters, except for dividends, which are represented by $d_t := D_t/Y$.

2.1 Non-Separable Utility function

Following Bilbiie (2020a), the period utility function in the model is as follows:

$$U(C,N) = \frac{1}{1-\xi/(1-\gamma)}\left(\frac{C^{1-\gamma}}{1-\gamma} - \Xi\frac{N^{1+\varphi}}{1+\varphi}\right)^{1-\xi/(1-\gamma)},$$

where C is consumption, N is hours worked, and Ξ is the parameter that adjusts the steady state of the hours worked. The parameters γ, φ, and ξ satisfy the following relationship:

$$\begin{aligned}
\gamma &= -\frac{U_{CC}C}{U_C} + \frac{U_{CN}C}{U_N},\\
\varphi &= \frac{U_{NN}N}{U_N} - \frac{U_{CN}N}{U_C},\\
\xi &= -\frac{U_{CN}C}{U_N}\left[1 + \frac{1-\gamma}{1+\varphi}\frac{N}{C}\frac{U_N}{U_C}\right].
\end{aligned}$$

When the wage rate (W_t) equals the elasticity of substitution between hours worked and consumption or $W_t = -U_N(C_t,N_t)/U_C(C_t,N_t)$, the log-linear approximation is as follows:

$$w_t = \varphi n_t + \gamma c_t. \tag{1}$$

The inverse of φ represents wage elasticity with respect to constant consumption. As noted in Bilbiie (2020a), this is not Frisch elasticity or wage elasticity with constant marginal utility. Parameter γ is the (relative) income effect on the labor supply and does not necessarily equal the reciprocal of the elasticity of intertemporal substitution (EIS).

I assume that $\gamma \geq 0$ and $\varphi > 0$. Then, The log-linear approximation of the marginal utility for consumption ($MU_t := U_C$) is

$$\begin{aligned} mu_t &= -(\gamma+\kappa)c_t + \frac{U_{CN}N}{U_C}n_t \\ &= -(\gamma+\kappa)c_t - \kappa\frac{N}{C}\frac{U_N}{U_C}n_t, \\ \kappa := -\frac{U_{CN}C}{U_N} &= \xi\left[1+\frac{1-\gamma}{1+\varphi}\frac{N}{C}\frac{U_N}{U_C}\right]^{-1}. \end{aligned}$$

Bilbiie (2020a) defines parameter κ as the degree of complementarity between consumption and hours worked. When $\kappa < 0$, the utility has substitutability between C and N.

When the economy is not distorted in the steady state, all labor income is used for consumption ($-(N/C)(U_N/U_C) = WN/C = 1$). In this economy,

$$\begin{aligned} mu_t &= -\gamma c_t - \kappa(c_t - n_t), \\ \kappa &= \frac{\xi(1+\varphi)}{\varphi+\gamma}. \end{aligned}$$

Clearly, $\kappa = 0 \Leftrightarrow \xi = 0$, and κ is proportional to ξ. Notably, $\gamma = 1$ implies $\xi = \kappa$. Throughout this study, I assume that the economy is non-distorted. Hereafter, both κ and ξ are used interchangeably as the degree of complementarity between consumption and hours worked.

I assume U to be concave. With $-(N/C)(U_N/U_C) = WN/C = 1$, the concavity condition is

$$\kappa \geq -\frac{\varphi\gamma}{\varphi+\gamma} \Leftrightarrow \xi \geq \frac{\varphi\gamma}{1+\varphi} \Leftrightarrow \gamma + \frac{1+\varphi}{\varphi}\xi \geq 0. \tag{2}$$

See Online Appendix for proof. Even in the case of substitutability ($\kappa < 0$), concavity can be established.

This functional form of the utility function nests the constant relative risk aversion- (CRRA), Greenwood–Hercowitz–Huffman- (GHH), and King–Plosser–Rebelo- (KPR) type forms. That is, with $\xi = 0$, the functional form is reduced to the CRRA-type one:[*1]

$$U(C, N) = \frac{1}{1-\gamma} C^{1-\gamma} - \frac{\Xi}{1+\varphi} N^{1+\varphi},$$

where the degree of complementarity is 0, $1/\varphi$ is the Frisch elasticity, and the income effect (γ) is the inverse of the EIS. When $\gamma = 0$, the functional form is reduced to GHH-type one:[*2]

$$U(C, N) = \frac{1}{1-\xi} \left(C - \frac{\Xi}{1+\varphi} N^{1+\varphi} \right)^{1-\xi},$$

in which the income effect is 0, indicating that the labor supply is determined by the wage rate alone. When $\gamma \to 1$, the functional form is reduced to KPR-type form:[*3]

$$U(C, N) = -\frac{1}{\xi} \left[C \exp \left\{ - \frac{\Xi N^{1+\varphi}}{1+\varphi} \right\} \right]^{-\xi},$$

where the income effect is unity, ensuring a balanced growth path.

Using the parameters γ, φ, and ξ, this functional form can separately control for the income effect, wage elasticity, and degree of complementarity.

2.2 Outline of TANK Model

I now proceed to the model descriptions. I consider several features that differ from those of the standard RANK model. See Online Appendix for the full descriptions.

The model includes two types of households: Type H (hand-to-mouth) and Type S (saver). The total population is normalized to be 1. The ratio of type H is $\lambda \in [0, 1]$, whereas that of type S is $1 - \lambda$. Type H households, which are subject to hand-to-mouth constraints, make optimal labor supply decisions to determine their income. Their income, denoted by Y_t^H, combines their wage rate (W_t), hours worked (N_t^H), and any transfers (TR_t^H). The remaining type S consumers receive labor income and profits from illiquid shares after taxes. The standard intratemporal optimality

[*1] The origin of this appraisal is that this function has a constant relative risk aversion.
[*2] This functional form was first proposed by Greenwood et al. (1988).
[*3] This functional form was first proposed by King et al. (1988).

condition determines the optimal number of hours worked, assuming identical elasticity across agents. This condition derives the aggregate labor supply function given by (1).

The model has the following two types of firms: final and intermediate good firms. The final goods market is perfectly competitive, whereas the intermediate goods market is monopolistically competitive. Substitution elasticity among the intermediate goods is denoted by ψ. A intermediate firm j sets the prices of their products, $P_t^I(j)$, subject to a quadratic price adjustment cost, as in Rotemberg (1982). The adjustment cost function[*4] is assumed to be

$$\frac{\eta}{2}\left(\frac{P_t^I(j) - P_{t-1}^I}{P_{t-1}^I}\right)^2,$$

where η is the degree of price adjustment, and P_t^I, the aggregate price level, is equal to the price of final goods P_t in a symmetric equilibrium. A subsidy policy of standard New Keynesian optimal sales is redistributive and taxes the shareholders of firms, resulting in a full-insurance steady state where the consumption of hand-to-mouth households (C^H) is equal to the consumption of non-hand-to-mouth households (C^S). Log-linearizing around this steady state, firms' profits vary inversely with real wage, denoted by $d_t = -w_t$.

The government implements both fiscal and monetary policies. The fiscal policy consists of an optimal subsidy policy, as discussed, and a redistribution policy. In the latter scheme, profits are taxed at rate τ and the proceeds are rebated lumpsum to hand-to-mouth households. The monetary policy controls nominal interest rates INT_t (based on 1) according to the following rule:

$$\begin{gathered} INT_t = \frac{\exp(-m_t)}{\beta}\mathbb{E}_t INFL_{t+1}^{\phi},\ \phi > 1, \\ m_t = \rho m_{t-1} + e_t,\ 0 < \rho < 1, \end{gathered}$$

[*4] Standard NK models typically employ the following price adjustment costs:

$$\frac{\eta}{2}\left(\frac{P_t^I(j) - P_{t-1}^I(j)}{P_{t-1}^I(j)}\right)^2.$$

Additionally, the log-linearization is identical to that of the Calvo case. When employing this adjustment cost, the result becomes more complex but remains robust. See Online Appendix for further details.

where $INFL_t = P_t/P_{t-1}$ refers to the rates of inflation (based on 1), and m_t is a zero-mean AR(1) monetary shock with a zero-mean independent and identically distributed random variables error ϵ_t.[*5]

Market clearing in the goods and labor market is summarized as total income $Y_t = C_t + (\eta/2)(INFL_t - 1)^2 Y_t$, total consumption $C_t = \lambda C_t^H + (1-\lambda)C_t^S$, and total hours worked $N_t = \lambda N_t^H + (1-\lambda)N_t^S$. With a full-insurance steady state, $Y = C = C^H = C^S = N = N^H = N^S = 1$, and log-linearizing around this steady state yields $y_t = c_t$, $c_t = \lambda c_t^H + (1-\lambda)c_t^S$, and $n_t = \lambda n_t^H + (1-\lambda)n_t^S$.

With a given aggregate income y_t, logarithmic consumption for each type is

$$\begin{aligned} c_t^H &= \chi y_t, \\ c_t^S &= \frac{1-\lambda\chi}{1-\lambda} y_t \end{aligned} \tag{3}$$

where $\chi := 1 + \varphi(1 - \tau/\lambda)$. Parameter χ denotes the elasticity of type H's consumption, which increases in λ and decreases in τ. Note that $\tau < \lambda$ is equivalent to $\chi > 1$. With $\chi > 1$, the elasticity of consumption of Type H is greater than unity or elastic, and that of Type S is less than unity or inelastic. With $\tau = 0$, $\chi = 1 + \varphi > 1$. When $\tau = \lambda$, or $\varphi = 0$, then $\chi = 1$ or $c_t = c_t^H = c_t^S = y_t$.

Since Campbell and Mankiw (1991, 1990, 1989) consider the case in which Type H consumes a constant fraction of aggregate income, this case is called the CM case. When $\lambda\chi < 1$, Type S consumption is procyclical with respect to y_t. The condition for $\lambda\chi > 1$ is $\varphi < (1-\lambda)/(\lambda - \tau)$. With a sufficiently large φ, the consumption of Type S may be countercyclical ($\lambda\chi < 1$) with respect to y_t.

Then, the logarithmic hours worked for each type is

$$\begin{aligned} n_t^H &= (1 + (1-\chi)\gamma/\varphi) y_t, \\ n_t^S &= \left(1 - \frac{\lambda(1-\chi)}{1-\lambda} \frac{\gamma}{\varphi}\right) y_t. \end{aligned} \tag{4}$$

[*5] The Taylor rule of a standard NK model is

$$INT_t = \frac{\exp(-m_t)}{\beta} INFL_t^{\phi_p} Y_t^{\phi_y}, \ \phi_p > 1.$$

The main results remain robust even after applying this rule. See Online Appendix for further details.

When $\chi > 1$, Type H's hours worked are inelastic, and those of Type S are elastic. When $\lambda = 0$ (RANK), $\tau = \lambda$ (CM), or $\gamma = 0$ (GHH), then $n_t = n_t^H = n_t^S = y_t$.

Then, the economic intuition is as follows. Let us examine the RANK model, in which a single agent works and receives all the profits. As aggregate income increases, demand also increases (owing to sticky prices), leading to an expansion in labor demand and an increase in real wages, which, in turn, decreases profits because wages represent marginal costs. As the same agent experiences both gains in labor income and losses in capital income, the income distribution between the two remains neutral.

The TANK model breaks this neutrality by introducing a general equilibrium feedback loop, wherein Type H's actions influence Type S's income through an income effect. First, I consider the case with no redistribution ($\tau = 0$). If demand increases for any reason, causing the real wage to rise (by moving along an upward-sloping labor supply curve, where $\varphi > 1$), Type H's income increases. Their demand also increases proportionally because they do not experience the negative income effect of declining profit. The result is an additional boost to aggregate demand, causing labor demand to shift even further, increasing wages, and so on. Ultimately, equilibrium is reached when Type S, whose income decreases as profits decline, responds optimally by increasing the hours worked to generate the additional demand needed to compensate for the income loss.

This mechanism is dampened when redistribution is introduced with $\tau > 0$. Type H consumers begin internalizing some of the negative income effect of declining profits through the transfer and, consequently, do not increase their demand as much. The case with $\chi = 1 \Leftrightarrow \tau = \lambda$ (CM) is achieved when profits are uniformly distributed, resulting in the income effect's disappearance. By contrast, when Type H receives an unequal share of the profits ($\tau > \lambda$), the opposite occurs.

The inequality measure is defined as the logarithmic consumption ratio of type S to type H:

$$gap_t := c_t^S - c_t^H = \frac{1-\chi}{1-\lambda} y_t. \tag{5}$$

When $\chi > 1$, the inequality measure is countercyclical. As the income of S fluctuates more than proportionally with the total income, inequality decreases during economic expansions and increases during economic recessions. Likewise, when $\chi < 1$, the inequality measure is procyclical. A more aggressive redistribution policy ($\tau > \lambda$) reverses the inequality gap.

3 Main Results

In this section, I present the main results and examine the impacts of heterogeneity and complementarity on the NK-IS curve, the NK consumption function, and the objective function of the optimal policy.

3.1 New Keynesian IS Curve

I now derive the three-equation NK model: IS curve, Phillips curve, and monetary policy (MP) equation. I first investigate the properties of the IS curve, and then discuss monetary multipliers in both the IS-MP analysis and the aggregate demand–aggregate supply (AD-AS) analysis cases.

Using

$$\begin{aligned}\kappa(c_t^S - n_t^S) &= \kappa\frac{\varphi+\gamma}{\varphi}\frac{\lambda(1-\chi)}{1-\lambda}y_t \\ &= \xi\frac{\varphi+1}{\varphi}\frac{\lambda(1-\chi)}{1-\lambda}y_t,\end{aligned}$$

I obtain the marginal utility of type S (mu_t^S) as follows:

$$\begin{aligned}mu_t^S &= -\left(\gamma c_t^S + \kappa(c_t^S - n_t^S)\right) \\ &= -\left(\gamma + \frac{\lambda(1-\chi)}{1-\lambda}\left\{\gamma + \xi\frac{(\varphi+1)}{\varphi}\right\}\right)y_t.\end{aligned} \tag{6}$$

With $\chi > 1$, the consumption of Type S is inelastic and that of hours worked is elastic ($c_t^S < n_t^S$). Thus, marginal utility increases with the degree of complementarity between consumption and hours worked. When $\lambda = 0$ (RANK) or $\chi = 1$ (CM), $mu_t^S = -\gamma y_t$, indicating that marginal utility is independent of complementarity. With $\xi = 0$ (CRRA-TANK),

$$mu_t^S = -\gamma\frac{1-\lambda\chi}{1-\lambda}y_t,$$

which is the case in Bilbiie's studies (2020b, 2008).

With $\gamma = 0$ (GHH-TANK),

$$mu_t^S = -\xi \frac{\varphi + 1}{\varphi} \frac{\lambda(1-\chi)}{1-\lambda} y_t.$$

Interestingly, unlike the GHH-RANK ($\gamma = \lambda = 0$), marginal utility depends on output.

Substituting (6) in the following Euler equation:

$$mu_t^S - \mathbb{E}_t \left[mu_{t+1}^S \right] = int_t - \mathbb{E}_t infl_{t+1},$$

I obtain the NK-IS curve,

$$y_t = \mathbb{E}_t \left[y_{t+1} \right] - \frac{int_t - \mathbb{E}_t infl_{t+1}}{\tilde{\gamma}} \tag{7}$$

where

$$\tilde{\gamma} = \gamma + \frac{\lambda(1-\chi)}{1-\lambda} \left\{ \gamma + \xi \frac{(\varphi+1)}{\varphi} \right\}. \tag{8}$$

The inverse of $\tilde{\gamma}$ is called the GE-EIS. Following Bilbiie (2020a), the GE-EIS is distinguished from the EIS by considering the general equilibrium effects. The EIS is the percentage change in consumption growth that can be attributed to a percentage change in the marginal rate of intertemporal substitution.[*6] In other words, the EIS is $\partial(c_{t+1} - c_t)/\partial(mu_t - mu_{t+1}) = 1/(\gamma + \kappa)$. In equilibrium, consumption is the output and the marginal rate of intertemporal substitution is the real interest rate $INT_t / \mathbb{E}_t INFL_{t+1}$. Therefore, the GE-EIS is $d(\mathbb{E}_t y_{t+1} - y_t)/dr_t = 1/\tilde{\gamma}$. In the RANK model, the EIS and GE-EIS have the same value $1/\gamma$.

Assume $\tilde{\gamma} > 0$, $\chi > 1$, and $\kappa > 0$. Then, (2) and (8) indicate $\tilde{\gamma} < \gamma$ or $1/\tilde{\gamma} > 1/\gamma$. Because $\tilde{\gamma}$ in (8) decreases with ξ and λ, the GE-EIS increases with the degree of complementarity ξ and ratio of hand-to-mouth households λ.

[*6] The marginal rate of intertemporal substitution is

$$MRIS_{t+1} = U_C(C_t, N_t)/\{\beta U_C(C_{t+1}, N_{t+1})\},$$

and the EIS is $\partial \ln(C_{t+1}/C_t)/\partial \ln MRIS_{t+1}$. As discussed later, the inverse of $MRIS$ is called the stochastic discount factor.

Let us consider the conditions for a positive GE-EIS. Equation (8) can be expressed as follows:

$$\tilde{\gamma} = \gamma\frac{1-\lambda\chi}{1-\lambda} + \xi\frac{\lambda(1-\chi)}{1-\lambda}\frac{(\varphi+1)}{\varphi}.$$

Therefore, the necessary and sufficient condition for $\tilde{\gamma} > 0$ is

$$\lambda\chi < 1 \ \& \ \gamma > \xi\frac{\lambda(\chi-1)}{1-\lambda\chi}\frac{\varphi+1}{\varphi}$$

or

$$\lambda\chi > 1 \ \& \ \gamma < \xi\frac{\lambda(\chi-1)}{1-\lambda\chi}\frac{\varphi+1}{\varphi}.$$

The latter case is pathological and is ignored in this study. The first case indicates that a sufficiently large γ is required with $\lambda\chi < 1$ and $\chi > 1$. In the case of the separable TANK ($\xi = 0$), only $\lambda\chi < 1$ is required for a positive GE-EIS, but $\chi > 1$ is also required for the GE-EIS to increase with λ, as discussed in Bilbiie (2008). However, in the case of the GHH-TANK ($\gamma = 0$) with $\chi < 1$, the GE-EIS is positive and decreases with λ.

Remember that $\chi > 1 \Leftrightarrow \lambda > \tau$ and $\lambda\chi < 1 \Leftrightarrow \varphi < (1-\lambda)/(\lambda-\tau)$. The above discussion can be summarized as follows:

> **Proposition** 1: Assume $\xi \geq 0$, $\lambda > \tau$, $\varphi < (1-\lambda)/(\lambda-\tau)$, and
>
> $$\gamma > \frac{\xi(\varphi+1)(\lambda-\tau)}{1-\lambda-\varphi(\lambda-\tau)}.$$
>
> Then, inequality is countercyclical ($\chi > 1$), the consumption of type S households is procyclical ($\lambda\chi < 1$), and the GE-EIS ($1/\tilde{\gamma}$) is positive and greater than $1/\gamma$. A higher degree of complementarity and a higher ratio of hand-to-mouth households generate a higher GE-EIS.

For example, when $\tau = 0$, $\varphi = 1$, $\lambda = 1/4$, and $0 \leq \xi < \gamma$, the assumption in the proposition above is established.

As Bilbiie (2020a) suggests, complementarity has no impact on the GS-EIS in a RANK economy. By contrast, complementarity, as well as the hand-to-mouth consumer portion, has a crucial effect even with a TANK economy. The countercyclical inequality and procyclical consumption of Type S both increase the degree of the GE-EIS. I intuit this finding as follows: I begin with a RANK model with separable utility ($\lambda = \kappa = 0$). In this case, the EIS is equivalent to the inverse of the income effect on labor supply γ. When γ is lower, a change in the marginal rate of the intertemporal substitution (or interest rate) precipitates a greater change in consumption today, relative to tomorrow. A smaller income effect implies that equilibrium income must increase by a relatively larger amount in order to achieve the same shift in the constant consumption labor supply. Therefore, with separable preferences, the intertemporal substitution and income/wealth effect on labor supply are linked in a one-to-one manner.

In the case of a TANK model ($\lambda > 0$), because Type H consumers do not have access to the bond market, the GE-EIS is determined by Type S's consumption and labor. As previously indicated, with countercyclical inequality ($\chi > 1$), Type H consumption is elastic vis-a-vis aggregate demand, whereas consumption in Type S is inelastic. The more limited the percentage of consumers accessible, the greater the impact on aggregate demand for changes in the interest rate.

With $\kappa \neq 0$, the income effect γ includes an additional factor beyond the consumption curvature that arises from complementarity. When $n_t = c_t$, complementarity does not impact the GE-EIS. However, if $n_t \neq c_t$, complementarity must compensate for this imbalance and prompt the necessary variation in working hours to achieve the consumption variation demanded by the intertemporal substitution. With countercyclical inequality ($\chi > 1$), Type S consumption is inelastic and labor is elastic, $n_t > c_t$, implying that complementarity enhances the GE-EIS.

The three-equation TANK model with non-separable utility comprises (7),

$$infl_t = \frac{\psi(\gamma + \varphi)}{\eta} y_t, \tag{9}$$

and

$$int_t = \phi \cdot \mathbb{E}_t infl_{t+1} - m_t. \tag{10}$$

The equation (9) is called the simplified Philips curve or aggregate supply (AS) curve, and the last one (10) is the Monetary Policy (MP) curve. Using these equations, I briefly explore monetary multipliers within two analytical frameworks: the IS-MP model and the AD-AS approach.

First, we conduct an IS-MP analysis. With $E_t y_{t+1} = \rho y_t$ $(0 < \rho < 1)$ and a fixed price $(infl_t = 0)$, the MP curve is $r_t = int_t = -m_t$.

Thus, the multiplier effect under fixed prices is as follows:

$$\Omega := \left.\frac{dy_t}{dm_t}\right|_{infl_t=0} = \frac{dy_t}{d(-r_t)} = \frac{1}{\tilde{\gamma}}\frac{1}{1-\rho} \quad .$$

With $\lambda = 0$ (RANK), I obtain $\Omega = 1/\{\gamma(1-\rho)\}$, which implies that complementarity does not matter; otherwise, with the countercyclical inequality measure $(\chi > 1)$, the larger the ratio of Type H consumers (λ) and the degree of complementarity (ξ), the larger the multiplier effect Ω.

Substituting (9) into (7) yields the following aggregate demand (AD) curve:

$$\begin{aligned} y_t &= \mathbb{E}_t\left[y_{t+1}\right] - \frac{(\phi-1)\mathbb{E}_t infl_{t+1} - m_t}{\tilde{\gamma}} \\ &= \left\{1 - \frac{(\phi-1)(\psi/\eta)\left(\gamma+\varphi\right)}{\tilde{\gamma}}\right\}\mathbb{E}_t\left[y_{t+1}\right] + \frac{1}{\tilde{\gamma}}m_t. \end{aligned}$$

When $\phi > 1$ and the GE-EIS $(1/\tilde{\gamma})$ is positive, the coefficient of $\mathbb{E}_t\left[y_{t+1}\right]$ is less than 1, and the equilibrium is deterministic.

Then, I conduct an AD-AS analysis with $E_t infl_{t+1} = \rho \cdot infl_t$ and $E_t y_{t+1} = \rho y_t$. With $\phi > 1$, the multiplier effect under price rigidity is as follows:

$$\frac{dy_t}{dm_t} = \frac{1}{\Omega^{-1} + \rho(\phi-1)(\psi/\eta)\left(\gamma+\varphi\right)} < \Omega.$$

This effect is smaller than that of Ω. The degree of complementarity (ξ or κ) has no impact other than Ω. In particular, the multiplier with flexible prices $(\psi \to \infty)$ is 0, or monetary policy does not affect output.

3.2 New Keynesian Cross

I follow Bilbiie (2020a) to derive an NK consumption function that considers complementarity and, as a result, measures the MPC.

In this framework, the multiplier under a fixed price is the total effect of monetary policy on consumption $\Omega = dc_t/d(-r_t)$. The effect Ω can be decomposed into two effects. The direct effect is the partial derivative $\partial c_t/\partial(-r_t)$. The indirect effect is the derivative along the path where $c_t = y_t$; however, the interest rate is fixed. That is,

$$\begin{aligned}\frac{dc_t}{d(-r_t)} &= \frac{\partial c_t}{\partial(-r_t)} + \frac{\partial c_t}{\partial y_t}\frac{dy_t}{d(-r_t)}\\ &= \frac{\partial c_t}{\partial(-r_t)} + \frac{\partial c_t}{\partial y_t}\frac{dc_t}{d(-r_t)}.\end{aligned}$$

Denoting $\Omega_D = \partial c_t/\partial(-r_t)$, $\Omega_I = \Omega - \Omega_D$, and $\omega = \partial c_t/\partial y_t$, I obtain:

$$\begin{aligned}\Omega &= \Omega_D/(1-\omega)\\ \omega &= \Omega_I/\Omega,\end{aligned}$$

where the relative share of the indirect effect ω is the MPC. To calculate the MPC, Bilbiie (2020a) proposes the NK cross, where the consumption function is expressed as a function of the current income for a given real interest rate:

$$c_t = \omega y_t - \Omega_D r_t,$$

where the slope ω is the MPC and the shift in Ω_D reflects changes in autonomous expenditure when policy changes occur. In models without capital and inventories, Ω_D is the same as the intertemporal substitution. Households tend to increase their consumption in the present when the interest rate decreases at a given income level. Income adjustments are necessary to achieve equilibrium because there are no assets to liquidate or "disinvest." This mechanism is nearly the same as that of the old Keynesian cross, in which the "ad-hoc" consumption function is a function of only the current income, and the investment function is included. Bilbiie (2020a) utilizes the intertemporal budget constraint to derive a consumption function with a micro-foundation. However, he considers the CRRA utility function, which is separable between consumption and hours worked.

As detailed in Online Appendix, the intertemporal budget constraint for agent j is

$$\sum_{i=0}^{\infty} Q^j_{t,t+i} C^j_{t+i} = \sum_{i=0}^{\infty} Q^j_{t,t+i} Y^{D,j}_{t+i}, \tag{11}$$

where $Y^{D,j}_t$ is the disposable income (the sum of labor and asset income) and $Q^j_{t,t+i}$ is a stochastic discount factor.

The first-order conditions at each date and state are

$$Q_{t,t+i}^{j} = \beta^{i} \frac{U_C(C_{t+i}^{j})}{U_C(C_t^{j})}, \tag{12}$$

where $0 < \beta < 1$. By log-linearizing the intertemporal budget constraint (11) and using the Euler equation and definition of stochastic discount factors (12), I obtain the consumption function in recursive form:

$$c_t^j = (1-\beta) y_t^{D,j} + \beta \left(\mathbb{E}_t c_{t+1}^j - \frac{1}{\gamma+\kappa} \left(r_t + \kappa \left(\mathbb{E}_t n_{t+1}^j - n_t^j \right) \right) \right). \tag{13}$$

Note that agent j takes as given r_t, n_t, and $y_t^{D,j}$.

First, I consider a non-separable RANK case. The consumption function for future consumption and current disposable income can be expressed in several ways. Bilbiie (2020a) assumes that

$$\mathbb{E}_t c_{t+1}^S = \frac{1-\lambda\chi}{1-\lambda} \mathbb{E}_t c_{t+1}$$
$$y_t^{D,S} = \frac{1-\lambda\chi}{1-\lambda} y_t$$

in the separable TANK case. That is, the expected variable value is attributed to future consumption, whereas the current variable is attributed to income. Following this rule, I assume $\mathbb{E}_t n_{t+1} = \mathbb{E}_t c_{t+1}$, and $n_t = y_t$.

Then, I obtain

$$\begin{aligned} c_t &= \frac{\beta\gamma}{\gamma+\kappa} \mathbb{E}_t c_{t+1} + \left\{ 1 - \frac{\beta\gamma}{\gamma+\kappa} \right\} y_t - \frac{\beta r_t}{\gamma+\kappa} \\ &= \tilde{\beta} \mathbb{E}_t c_{t+1} + (1-\tilde{\beta}) y_t - \frac{\tilde{\beta} r_t}{\gamma} \end{aligned}$$

where $\tilde{\beta} = \beta\gamma/(\gamma+\kappa)$. If $\tilde{\beta}$ is replaced with β, the non-separable RANK is the same as the separable RANK, which involves an identification problem. Notably, by imposing a good market clearing $c_t = y_t$, I obtain the familiar RANK-IS curve:

$$y_t = E_t y_{t+1} - r_t/\gamma.$$

With $\mathbb{E}_t c_{t+1} = \rho c_t$, I obtain the consumption function:

$$c_t = \frac{1-\tilde{\beta}}{1-\tilde{\beta}\rho} y_t + \frac{1}{\gamma} \frac{\tilde{\beta}}{1-\tilde{\beta}\rho} r_t.$$

The coefficient of the first term on the right-hand side is the MPC with non-separable (ω). For GHH ($\gamma = 0$), the consumption function reduces to $c_t = y_t + r_t$, and the MPC is unity.

Assume that the degree of complementarity is positive or $\kappa > 0$. Then,

$$\omega = \frac{1-\tilde{\beta}}{1-\tilde{\beta}\rho} = 1 - \frac{(1-\rho)\beta}{\kappa/\gamma + 1 - \beta\rho} > \frac{1-\beta}{1-\beta\rho},$$

implying that the MPC increases as the degree of complementarity increases. The necessary and sufficient condition for $0 < \omega < 1$ is

$$0 < \tilde{\beta} < 1 \Leftrightarrow 0 < \beta < 1 \ \& \ \kappa > \beta - 1,$$

implying that a positive complementarity ensures $0 < \omega < 1$.

Next, I consider a non-separable TANK case. Assuming

$$\begin{aligned}
&\mathbb{E}_t c^S_{t+1} = \frac{1-\lambda\chi}{1-\lambda}\mathbb{E}_t c_{t+1}, \mathbb{E}_t n^S_{t+1} = \left(1 - \frac{\lambda(1-\chi)}{1-\lambda}\frac{\gamma}{\varphi}\right)\mathbb{E}_t c_{t+1}, \\
&y^{D,S}_t = \frac{1-\lambda\chi}{1-\lambda} y_t, \ n^S_t = \left(1 - \frac{\lambda(1-\chi)}{1-\lambda}\frac{\gamma}{\varphi}\right) y_t,
\end{aligned}$$

equation (13) for Type S is

$$\begin{aligned}
c^S_t = &\frac{\tilde{\beta}}{1-\lambda}\left\{1 - \lambda\chi + \kappa\lambda(1-\chi)\left(\frac{1}{\varphi} + \frac{1}{\gamma}\right)\right\}\mathbb{E}_t c_{t+1} \\
&+ \frac{1}{1-\lambda}\left\{(1-\tilde{\beta})(1-\lambda\chi) - \tilde{\beta}\kappa\lambda(1-\chi)\left(\frac{1}{\varphi} + \frac{1}{\gamma}\right)\right\} y_t - \frac{\tilde{\beta} r_t}{\gamma}.
\end{aligned}$$

See Online Appendix for a detailed derivation. Therefore, using $c^H_t = \chi y_t$, I obtain the aggregate dynamic consumption function as

$$\begin{aligned}
c_t &= (1-\lambda)c^S_t + \lambda c^H_t \\
&= \frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma} E_t c_{t+1} + \left\{1 - \frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma}\right\} y_t - \frac{(1-\lambda)\tilde{\beta} r_t}{\gamma},
\end{aligned}$$

where $\tilde{\gamma}$ is defined in Equation (8). See Online Appendix. By imposing good market clearing $c_t = y_t$, I obtain the separable TANK-IS curve (7).

With $\mathbb{E}_t c_{t+1} = \rho c_t$, I obtain the following proposition:

Proposition 2: In a non-separable TANK model, aggregate MPC ω is

$$\begin{aligned}\omega &= \frac{1-(1-\lambda)\tilde{\beta}\tilde{\gamma}/\gamma}{1-(1-\lambda)\tilde{\beta}\rho\tilde{\gamma}/\gamma} \\ &= \frac{1-\tilde{\beta}\left\{1-\lambda\chi-\xi\lambda(\chi-1)\frac{\varphi+1}{\gamma\varphi}\right\}}{1-\tilde{\beta}\rho\left\{1-\lambda\chi-\xi\lambda(\chi-1)\frac{\varphi+1}{\gamma\varphi}\right\}}.\end{aligned}$$

Assume that the conditions in Proposition 1 are satisfied. Then, $0 < \omega < 1$, and ω increases with the degree of complementarity and ratio of hand-to-mouth households.

See Online Appendix for proof.

I now provide some comments on this proposition. When $\lambda = 0$, ω is reduced to the MPC of the non-separable RANK model: $(1-\tilde{\beta})/(1-\tilde{\beta}\rho)$, with $\tilde{\beta} = \beta\gamma/(\gamma+\kappa)$. When $\xi = 0$ (CRRA-TANK), it reduces to

$$\omega = \frac{1-\beta(1-\lambda\chi)}{1-\beta(1-\lambda\chi)\rho}.$$

When Type S consumption is procyclical ($\lambda\chi < 1$), $0 < \omega < 1$, as discussed in Bilbiie (2020b). Interestingly, ω is independent of γ. When $\gamma = 0$ (GHH-TANK), the MPC reduces to

$$\omega = \frac{1-\beta\lambda(1-\chi)}{1-\beta\lambda(1-\chi)\rho}.$$

Then, the condition for $0 < \omega < 1$ is $\chi < 1$ or the inequality measure is procyclical.

In conjunction with the discussion in the previous subsection, this proposition suggests that when consumption and hours worked are complementary ($\xi > 0$), inequality is countercyclical ($\chi > 1$) with aggregate demand y_t, the consumption of Type S households is procyclical ($\lambda\chi < 1$) with y_t, and the income effect γ is sufficiently large. Then, the MPC ω is greater than 0 and less than unity, and the GE-EIS is positive. Furthermore, the degree of complementarity ξ and the ratio of hand-to-mouth households λ increase both values and multiplier Ω, amplifying the effect of monetary policy.

3.3 Optimal Monetary Policy

In this subsection, I explore the optimal monetary policy in the case of a non-separable TANK model. First, we derive the objective function using a second-order log approximation, demonstrating that the objective function of the TANK model differs from that of the RANK model only in terms of weights. Next, I briefly discuss an optimal policy.

The second-order log approximation of the objective function of Type j is derived as follows:

$$
\begin{aligned}
\frac{U_t^j - U^j}{U_C^j C^j} =& c_t^j + \frac{1-\gamma}{2}(c_t^j)^2 + \frac{N^j}{C^j}\frac{U_N^j}{U_C^j}\left(n_t^j + \frac{1+\varphi}{2}n_t^2\right) \\
& - \frac{1}{2}\kappa^j \left(c_t^j + \frac{N^j}{C^j}\frac{U_N^j}{U_C^j}n_t^j\right)^2, \\
\kappa^j =& - U_{CN}^j C^j / U_N^j
\end{aligned}
$$

Consider the case of the RANK model with no distorted economy. The above is

$$
\frac{U_t - U}{U_C C} = c_t + \frac{1-\gamma}{2}c_t^2 - \left(n_t + \frac{1+\varphi}{2}(n_t)^2\right) - \frac{1}{2}\kappa\left(c_t - n_t\right)^2.
$$

Whether or not $\kappa = 0$, the objective function remains unchanged because the last term vanishes at equilibrium ($c_t = n_t$).

Next, consider the case of the TANK model in which $C = C^j, N = N^j, U = U^j, U_C = U_C^j$, and $\kappa^j = \kappa$. Then, the total objective function is

$$
\begin{aligned}
& \lambda\frac{U_t^H - U}{U_C C} + (1-\lambda)\frac{U_t^S - U}{U_C C} \\
& = -\eta\frac{infl_t^2}{2} - \frac{\gamma}{2}\{\lambda(c^H)^2 + (1-\lambda)(c^S)^2\} \\
& - \frac{\varphi}{2}\{\lambda(n_t^H)^2 + (1-\lambda)(n_t^S)^2\} \\
& - \frac{\kappa}{2}\{\lambda(c^H - n_t^H)^2 + (1-\lambda)\left(c_t^S - n_t^S\right)^2\}.
\end{aligned}
$$

See Online Appendix for a detailed derivation. Remember that (3) and (4). The first and second terms are expressed as follows:

$$-\frac{\eta}{2}infl_t^2 - \frac{\gamma+\varphi}{2}\left(1+\frac{\gamma}{\varphi}\frac{\lambda(\chi-1)^2}{1-\lambda}\right)y_t^2,$$

The above is the same as the welfare function in Bilbiie (2024).

The final term is as follows:

$$-\frac{\kappa}{2}\frac{\sigma+\varphi}{\varphi}\frac{\lambda(\chi-1)^2}{1-\lambda}y_t^2 = -\frac{1}{2}\xi\frac{\varphi+1}{\varphi}\frac{\gamma+\varphi}{\varphi}\lambda(1-\lambda)gap_t^2,$$

where gap_t is defined in (5). Complementarity and the size of gap cause the output to be heavily weighted. Whether the inequality measure is procyclical or countercyclical does not matter here.

In summary, I obtain the following proposition:

> **Proposition 3**: Solving the welfare maximization problem is equivalent to minimizing the following objective function:
>
> $$\min \frac{1}{2}E_0\sum_{t=0}^{\infty}\{infl_t^2 + \alpha y_t^2\} \tag{14}$$
>
> where $\alpha := \alpha_y\tilde{\alpha}$, $\alpha_y := (\gamma+\varphi)/\eta$, and
>
> $$\begin{aligned}\tilde{\alpha} &:= \left(1+\left\{\gamma+\xi\frac{\varphi+1}{\varphi}\right\}\frac{\lambda(\chi-1)^2}{\varphi(1-\lambda)}\right)\\ &= \left(1+\left\{\gamma+\xi\frac{\varphi+1}{\varphi}\right\}\frac{\lambda(1-\lambda)}{\varphi}\frac{gap_t^2}{y_t^2}\right).\end{aligned}$$

When $\lambda = 0$ (RANK) or $\chi = 1$ (CM-TANK), $\alpha = \alpha_y$. When $\xi = 0$ (Separable TANK), $\alpha y_t^2 = \alpha_y y_t^2 + \lambda(1-\lambda)(\gamma/\varphi)gap_t^2$.

Recall that (2) is the necessary and sufficient condition for a concave utility function. As long as $\xi \geq 0$, the objective function in the non-separable TANK model (14) is weighted more toward output than toward inflation compared with the RANK model ($\alpha > \alpha_y$). The fewer people involved in the bond market, the smaller the weight of inflation.

Now, I examine the optimal policies under "discretion" and "commitment." To implement the discretion policy, equation (14) is solved under the assumption that the central bank lacks commitment and treats expectations as fixed parameters rather than accounting for the impact of its actions on them. As a result, the central bank re-optimizes each period, subject to (9), with expectations fixed at the decision time. Because this problem is mathematically equivalent to that in the RANK economy, I obtain the following solution:

$$y_t = -\frac{(\psi/\eta)(\gamma+\varphi)}{\alpha} infl_t = -\frac{\psi}{\tilde{\alpha}} infl_t .$$

To achieve the optimal (timeless) commitment policy, one must adopt a different targeting rule, such as the approach in the RANK economy discussed in Woodford (2003), on the time-inconsistent Ramsey equilibrium. Normalizing the initial (log-linearized) price level to 0, I obtain the following solution:

$$y_t = -\frac{\psi}{\tilde{\alpha}} \sum_{j=0}^{t} infl_{t-j} = -\frac{\psi}{\tilde{\alpha}} p_t$$

The optimal commitment policy requires targeting the price level.

4 Conclusion

This study uses a TANK model with non-separable preferences to analyze how monetary policy affects the economy. The complementarity between consumption and hours worked significantly contributes in the GE-EIS, MPC, and objective function for optimal monetary policy. When inequality is countercyclical with aggregate demand, both the GE-EIS and MPC increase in the degree of complementarity and ratio of hand-to-mouth households. Irrespective of whether inequality is procyclical or countercyclical, a higher degree, a higher ratio, and a larger size of inequality gap cause the weight of output in the objective function to be larger than that of inflation.

This study's findings highlight the importance of incorporating complementarity into the building models. Most dynamic stochastic general equilibrium models assume separable preferences; however, accounting for complementarity can enhance the accuracy of policy effect predictions. The next step is to conduct a Bayesian estimation for medium-scale dynamic stochastic general equilibrium models that incorporate separable preferences and capital stocks.

Concerning theoretical development, Bilbiie (2008) conducted a theoretical study on the possibility of an inverted IS curve, while I assume the elasticity of intertemporal substitution to be positive for valid marginal propensity to consume. Further exploration of this inversion will yield meaningful insights. We may also incorporate complementarity into various economic models and analyze them analytically. As such, the THANK model (Bilbiie, 2024) is a priority, and it incorporates idiosyncratic shocks and provides a clearer economic intuition than the TANK model.

Finally, I would like to further clarify the relationship of the findings herein with the study by Auclert et al. (2023) (see section 1). A key challenge for future research is to examine in detail the connection between their adopted utility function and that employed in this study. Additionally, results of the analytical research on the correlation between price and wage rigidity—conducted by Bilbiie and Trabandt (2025)—could serve as a foundation for further exploring the relationship with the utility function in this study.

Appendix: Proof of Proposition 2

Let $f(x) = (1-x)/(1-\rho x)$ with $0 < \rho < 1$. If $0 < x < 1$, then $< f(x) < 1$. Furthermore,

$$\begin{aligned} f'(x) &= -1/(1-\rho x) - (1-x)\rho/(1-\rho x)^2 \\ &= \frac{\rho - 1 + \rho x(1-\rho)}{(1-\rho x)^2} = \frac{(1-\rho)(\rho x - 1)}{(1-\rho x)^2}. \end{aligned}$$

Thus, when $0 < x < 1$, then $f'(x) < 0$ or $f(x)$ is decreasing.

Now

$$\begin{aligned} \omega &= \frac{1-(1-\lambda)\tilde{\beta}\tilde{\gamma}/\gamma}{1-(1-\lambda)\rho\tilde{\beta}\tilde{\gamma}/\gamma} = f(x), \\ x &= (1-\lambda)\tilde{\beta}\tilde{\gamma}/\gamma. \end{aligned}$$

Since $0 < \tilde{\gamma} < \gamma$, $0 < \tilde{\beta} < \beta < 1$, $0 \leq \lambda < 1$, then $0 < (1-\lambda)\tilde{\beta}\tilde{\gamma}/\gamma < 1$. Thus $0 < \omega < 1$, and ω decreases with $(1-\lambda)\tilde{\beta}\tilde{\gamma}/\gamma$. Remember $\tilde{\beta}$ decreases with ξ, and $\tilde{\gamma}$

decreases with λ and ξ. Hence ω is increasing with λ and ξ.

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Online Appendix for “TANK Model with Non-Separable Preferences”

March 10, 2025

1 Concavity condition

For U to be concave, the determinant of the Hessian matrix must be non-negative:

$$\frac{\partial^2 \tilde{U}}{\partial C^2}\frac{\partial^2 \tilde{U}}{\partial (-N)^2} - \left(\frac{\partial^2 \tilde{U}}{\partial C \partial (-N)}\right)^2 = U_{CC}U_{NN} - U_{CN}^2 \geq 0.$$

Therefore both the transformed utility function and the original one have the same concavity condition.

Under the condition:

$$-\frac{N}{C}\frac{U_N}{U_C} = \frac{WN}{C} = 1,$$

then

$$\begin{aligned}
\gamma &= -\frac{CU_{CC}}{U_C} + \kappa \\
\varphi &= \frac{NU_{NN}}{U_N} - \kappa.
\end{aligned}$$

Therefore, the concavity condition can be rewritten as follows:

$$\begin{aligned}
&U_{CC}U_{NN} - U_{CN}^2 \geq 0 \\
\Leftrightarrow &\frac{CU_{CC}}{U_C}\frac{NU_{NN}}{U_N} - \frac{CNU_{CN}^2}{U_CU_N} \leq 0 \\
\Leftrightarrow &\frac{CU_{CC}}{U_C}\frac{NU_{NN}}{U_N} - \frac{NU_N}{CU_C}\left(-\frac{CU_{CN}}{U_N}\right)^2 \\
&= (-\gamma - \kappa)(\varphi + \kappa) + \kappa^2 \\
&= -(\gamma + \kappa)\varphi - \gamma\kappa \\
&= -\gamma\varphi - \kappa(\varphi + \gamma) \leq 0 \\
\Leftrightarrow &\kappa \geq -\frac{\gamma\varphi}{\varphi + \gamma}.
\end{aligned}$$

Since $\kappa = \xi(1 + \varphi)/(\varphi + \gamma)$, this condition is equivalent to

$$\xi \geq -\frac{\varphi\gamma}{1 + \varphi}.$$

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2 Full Model description of non-separable TANK model

2.1 Households

There are two types of households: type H (hand-to-mouse) and type S (saver) households. The total population is normalized to be 1. The ratio of type H is λ, white that of type S is $1-\lambda$.

Type S households

An agent S chooses consumption C_t^S, bond holdings B_t, and hours worked N_t^S solving standard intertemporal problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t^S, N_t^S)$$

subject to the sequence of the budget constraints:

$$C_t^S + \frac{B_t}{1-\lambda} + \frac{share_t^S}{1-\lambda} V_t + TX_t^S \le W_t N_t^S + \frac{INT_{t-1}}{INFL_t} \frac{B_{t-1}}{1-\lambda} + \frac{share_{t-1}^S}{1-\lambda}(V_t + D_t). \tag{1}$$

where C_t^S is consumption, N_t^S is hours worked, B_t is bond holdings, D_t is dividends, V_t is post-dividend stock, $share_t^S$ is a share of V_t, W_t is rate of wage, INT_{t-1} is rate of interest, $INFL_t$ is rate of inflation, and TX_t^S is taxation.

FOCs are given by:

$$MU_t^S = U_C\left(C_t^S,\ N_t^S\right) \tag{2}$$

$$MU_t^S W_t = -U_N\left(C_t^S,\ N_t^S\right) \tag{3}$$

$$1 = \mathbb{E}_t\left[\beta \frac{MU_{t+1}}{MU_t} \frac{INT_t}{INFL_{t+1}}\right] \tag{4}$$

$$V_t = \mathbb{E}_t\left[\sum_{i=1}^{\infty} \beta^i \frac{MU_{t+i}}{MU_t} D_{t+i}\right]$$

where MU_t is marginal utility with respect to consumption.

Type H households

Type H households cannot access to the asset market and do not smooth their consumption. In each period, all labor income is consumed; thus, the budget constraint is given by

$$C_t^H \le W_t N_t^H + TR_t^H. \tag{5}$$

where C_t^H is consumption and N_t^H is hours worked, and TR_t^H is a lump-sum transfer. The utility function of type H is the same as that of type S household:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t^H, N_t^H)$$

Then the first-order condition is given by:

$$W_t = -\frac{U_N(C_t^H, N_t^H)}{U_C(C_t^H, N_t^H)} \tag{6}$$

2.2 Firms

Final goods Firm

The final good market is perfectly competitive. The final good firm produces a final good Y_t using intermediate good $Y_t^{I,j}$. The production function is given by

$$Y_t = \left(\int_0^1 (Y_t^{I,j})^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}}, \ \psi > 1$$

where ψ is the elasticity of substitution among intermediate goods. Letting P_t and $P_t^{I,j}$ denote the prices of final and intermediate good respectively, the profit D_t^F is defined by

$$D_t^F := P_t Y_t - \int_0^1 P_t^{I,j} Y_t^{I,j} dj,$$

and the first-order condition of the profit maximization problem is

$$Y_t^{I,j} = Y_t \left(\frac{P_t^{I,j}}{P_t} \right)^{-\psi}. \tag{7}$$

Notice that the maximized profit is $D_t^F = 0$, and the price of final good is

$$P_t = \left[\int_0^1 (P_t^{I,j})^{1-\psi} dj \right]^{\frac{1}{1-\psi}}.$$

Intermediate goods Firm

The intermediate good market is monopolistically competitive. The intermediate good firm indexed by j produces a differentiated intermediate good $Y_t^{I,j}$ using labor input N_t^j. The production function is given by

$$Y_t^{I,j} = F(N_t^j).$$

Letting τ^S be the rate of subsidies and TX_t^I the lump-sum tax, the profit D_t^j is defined by

$$D_t^j = (1+\tau^S) \frac{P_t^{I,j}}{P_t} Y_t^{I,j} - f_P \left(\frac{P_t^{I,j}}{P_{t-1}} \right) Y_t - W_t N_t^j - TX_t^I$$

and the profit maximization problem subject to (7).

In symmetric equilibrium ($D_t^j = D_t$, $P_t^{I,j} = P_t^I = P_t$, $Y_t^{I,j} = Y_t^I = Y_t$), FOCs are summarized as:

$$\begin{aligned}
W_t =& MC_t F_N(N_t) \\
Y_t =& F(N_t) \\
INFL_t f_P'(INFL_t) =& \psi \left(MC_t - (1+\tau^S)(1 - 1/\psi_P) \right) && (8) \\
D_t =& (1 + \tau^S - f_P(INFL_t)) Y_t - W_t N_t - TX_t^I && (9)
\end{aligned}$$

where $INFL_t = P_t / P_{t-1}$ is the rate of inflation.

2.3 Government

Fiscal Policy

$$TX_t^I = \tau^S Y_t$$
$$\tau^S = \frac{1}{\psi - 1}$$

Substituting the above two equations into equations (8) and (9) leads to:

$$\begin{aligned} INFL_t f_P'(INFL_t) =& \psi (MC_t - 1) \\ D_t =& (1 - f_P(INFL_t))Y_t - W_t N_t. \end{aligned}$$

The former consists of a simple endogenous redistribution scheme: taxing profits at rate τ and rebating the proceedings lump-sum to consumer H:

$$\lambda TR_t^H = (1-\lambda) TX_t^S$$
$$(1-\lambda) TX_t^S = \tau D_t.$$

Monetary Policy

Monetary polity controls nominal interest rates INT_t following the rule:

$$INT_t = \frac{\exp(-m_t)}{\beta} \mathbb{E}_t INFL_{t+1}^{\phi}, \ \phi > 1$$
$$m_t = \rho m_{t-1} + e_t, \ 0 < \rho < 1,$$

where m_t is a zero-mean AR(1) monetary shock with a zero-mean iid error e_t

Market Clearing and Aggregate Conditions

Market clearing and aggregate conditions are as follows:

$$C_t = \lambda C_t^H + (1-\lambda) C_t^S$$
$$N_t = \lambda N_t^H + (1-\lambda) N_t^S$$

$$B_t = 0$$
$$share_t = 1$$

Substituting into (9) leads to:

$$D_t = (1 - f_P(INFL_t))Y_t - W_t N_t.$$

$$C_t^S + TX_t^S = W_t N_t^S + \frac{D_t}{1-\lambda}.$$

Substituting the above into (1) leads to:

$$\begin{aligned} C_t &= \lambda(W_t N_t^H + TR_t^H) + (1-\lambda)(W_t N_t^S - TX_t^S) + D_t \\ &= W_t N_t + D_t = (1 - f_P(INFL_t))Y_t. \end{aligned}$$

Therefore, the budget constraints is:

$$Y_t = C_t + f_P(P_t/P_{t-1}) Y_t.$$

2.4 Model Description

Functional Specifications:

$$
\begin{aligned}
U(C,N) &= \frac{1}{1-\xi/(1-\gamma)}\left(\frac{C^{1-\gamma}}{1-\gamma} - \Xi\frac{N^{1+\varphi}}{1+\varphi}\right)^{1-\xi/(1-\gamma)} \\
F(N) &= N \\
f_P(INFL_t) &= \frac{\eta}{2}\left(INFL_t - 1\right)^2
\end{aligned}
$$

Model summary (Nonlinear)

Using above functional forms, and ignoring V_t , we obtain the following (non-linear) model economy:

- Type S Household:

$$
\begin{aligned}
W_t &= \Xi\left(N_t^S\right)^{\varphi}\left(C_t^S\right)^{\gamma} \\
1 &= \mathbb{E}_t\left[\beta\frac{MU_{t+1}^S}{MU_t^S}\frac{INT_t}{INFL_{t+1}}\right] \\
MU_t^S &= [C_t^S]^{-\gamma}\left(\frac{[C_t^S]^{1-\gamma}}{1-\gamma} - \Xi\frac{[N_t^S]^{1+\varphi}}{1+\varphi}\right)^{-\xi/(1-\gamma)}
\end{aligned}
$$

- Type H Household:

$$
\begin{aligned}
C_t^H &= W_t + N_t^H + \frac{\tau}{\lambda}D_t \\
W_t &= \Xi\left(N_t^H\right)^{\varphi}\left(C_t^H\right)^{\gamma}
\end{aligned}
$$

- Firm:

$$
\begin{aligned}
W_t &= MC_t \\
\eta \cdot INFL_t\left(INFL_t - 1\right) &= \psi\left(MC_t - 1\right) \\
D_t &= (1 - \frac{\eta}{2}\left(INFL_t - 1\right)^2)Y_t - W_tN_t \\
Y_t &= N_t
\end{aligned}
$$

- Market Clearing Conditions & Monetary Policy:

$$
\begin{aligned}
C_t &= \lambda C_t^H + (1-\lambda)C_t^S \\
N_t &= \lambda N_t^H + (1-\lambda)N_t^S
\end{aligned}
$$

$$
\begin{aligned}
Y_t &= C_t + \frac{\eta}{2}\left(INFL_t - 1\right)^2 Y_t \\
INT_t &= \frac{\exp(-m_t)}{\beta}E_tINFL_{t+1}^{\phi} \\
m_t &= \rho m_t + e_t
\end{aligned}
$$

Steadystate

With $INFL = \Xi = 1$, the steady state is the followings:

$$
C = C^S = C^H = N = N^S = N^H = Y = W = 1
$$

$$
INT = 1/\beta
$$

Model summary (loglinear)

We denote by small letter log deviations from steady-state, except for dividends $d_t := D_t/Y$. Denoting by $r_t := int_t - E_t infl_{t+1}$ and eliminating W_t, we obtain the following (log-linear) model economy:

- Type S Household:

$$\begin{aligned} w_t &= \gamma c_t^S + \varphi n_t^S \\ r_t &= mu_t^S - \mathbb{E}_t \left[mu_{t+1}^S \right] \\ mu_t^S &= -(\gamma + \kappa) c_t^S + \kappa n_t^S \end{aligned}$$

- Type H Household:

$$\begin{aligned} c_t^H &= w_t + n_t^H + \frac{\tau}{\lambda} d_t \\ w_t &= \gamma c_t^H + \varphi n_t^H \end{aligned}$$

- Firm:

$$\begin{aligned} infl_t &= (\psi/\eta) w_t \\ d_t &= -w_t \end{aligned}$$

- Monetary Policy:

$$\begin{aligned} r_t &= (\phi - 1) E_t infl_{t+1} - m_t \\ m_t &= \rho m_t + e_t^m \end{aligned}$$

- Market Clearing Conditions:

$$\begin{aligned} y_t &= c_t = n_t \\ c_t &= \lambda c_t^H + (1-\lambda) c_t^S \\ n_t &= \lambda n_t^H + (1-\lambda) n_t^S \end{aligned}$$

2.5 Log-Linear Approximation to model

- Model Specification:

$$\begin{aligned} F(N) &= N \\ f_P(P/P_{t-1}) &= \frac{\eta}{2} \left(INF_t - 1 \right)^2 \end{aligned}$$

- Steady State: $D = 0$, $INF = 1/\beta$,

$$C = C^H = C^S = N = N^H = N^S = Y = W = INF = 1$$

- Log-Linear Transformation (eliminating mc_t)

– Type S Household:

$$\begin{aligned}
mu_t^S &= -(\gamma+\kappa)c_t^S + \kappa n_t^S \\
r_t &= mu_t^S - \mathbb{E}_t\left[mu_{t+1}^S\right] \\
w_t &= \gamma c_t^S + \varphi n_t^S
\end{aligned}$$

– Type H Household:

$$\begin{aligned}
c_t^H &= w_t + n_t^H + \frac{\tau}{\lambda} d_t \\
w_t &= \gamma c_t^H + \varphi n_t^H
\end{aligned}$$

– Firm:

$$\begin{aligned}
infl_t &= (\psi/\eta) w_t \\
d_t &= -w_t
\end{aligned}$$

– Market Clearing Conditions & Monetary Policy:

$$\begin{aligned}
c_t &= \lambda c_t^H + (1-\lambda)c_t^S \\
n_t &= \lambda n_t^H + (1-\lambda)n_t^S
\end{aligned}$$

$$\begin{aligned}
y_t &= c_t = n_t \\
r_t &= (\phi-1)E_t infl_{t+1} - m_t
\end{aligned}$$

2.6 Analytical Solution

Substituting $n_t^H = (w_t - \gamma c_t^H)/\varphi$ and $d_t = -w_t$ into $c_t^H = w_t + n_t^H + \frac{\tau}{\lambda} d_t$ leads to

$$\begin{aligned}
c_t^H &= w_t + (w_t - \gamma c_t^H)/\varphi - \frac{\tau}{\lambda} w_t \\
&= \frac{1+\varphi(1-\tau/\lambda)}{1+\gamma/\varphi}\frac{w_t}{\varphi}.
\end{aligned}$$

Substituting $w_t = \gamma c_t + \varphi n_t$ and $y_t = c_t = n_t$ into the above leads to

$$\begin{aligned}
c_t^H &= \frac{1+\varphi(1-\tau/\lambda)}{1+\gamma/\varphi}\frac{\gamma c_t + \varphi n_t}{\varphi} \\
&= \{1+\varphi(1-\tau/\lambda)\} y_t = \chi y_t.
\end{aligned}$$

Since $c_t = \lambda c_t^H + (1-\lambda)c_t^S$,

$$c_t^S = \frac{c_t - \lambda c_t^H}{1-\lambda} = \frac{1-\lambda\chi}{1-\lambda} y_t.$$

Similarly, substituting $c_t^H = (w_t - \varphi n_t^H)/\gamma$ and $d_t = -w_t$ into $n_t^H = -w_t + c_t^H - \frac{\tau}{\lambda} d_t$ leads to

$$\begin{aligned}
n_t^H &= w_t + (w_t - \varphi n_t^H)/\gamma - \frac{\tau}{\lambda} w_t \\
&= \frac{1+\varphi(1-\tau/\lambda)}{1+\varphi/\gamma}\frac{w_t}{\gamma}.
\end{aligned}$$

Substituting $w_t = \gamma c_t + \varphi n_t$ and $y_t = c_t = n_t$ into the above leads to

$$\begin{aligned} n_t^H &= \frac{1+\gamma(1-\tau/\lambda)}{1+\varphi/\gamma}\frac{\gamma c_t + \varphi n_t}{\gamma} \\ &= \{1+\gamma(1-\tau/\lambda)\}y_t = \{1+(1-\chi)\gamma/\varphi\}y_t. \end{aligned}$$

Since $n_t = \lambda n_t^H + (1-\lambda)n_t^S$,

$$n_t^S = \frac{n_t - \lambda n_t^H}{1-\lambda} = \left(1 - \frac{\lambda(1-\chi)}{1-\lambda}\frac{\gamma}{\varphi}\right) y_t.$$

Remains are followings:

$$\begin{aligned} w_t &= (\gamma+\varphi)y_t \\ d_t &= -(\gamma+\varphi)y_t \\ mc_t^S &= -\gamma c_t^S - \kappa(c_t^S - n_t^S) \\ r_t &= mu_t^S - \mathbb{E}_t\left[mu_{t+1}^S\right] \\ infl_t &= (\psi/\eta)(\varphi+\gamma)y_t \\ r_t &= (\phi-1)E_t infl_{t+1} - m_t \end{aligned}$$

3 Another type of NK models

Consider another formulation:

$$\begin{aligned} y_t &= \mathbb{E}_t\left[y_{t+1}\right] - \frac{int_t - \mathbb{E}_t infl_{t+1}}{\tilde{\gamma}}, \\ infl_t &= \frac{\psi(\gamma+\varphi)}{\eta}y_t + \beta\mathbb{E}_t infl_{t+1}, \\ int_t &= \phi_p \cdot infl_t + \phi_y y_t - m_t, \\ m_t &= \rho m_{t-1} + e_t^m, \end{aligned}$$

where

$$\begin{aligned} \tilde{\gamma} &:= \gamma + \frac{\lambda(1-\chi)}{1-\lambda}\left\{\gamma + \xi\frac{(\varphi+1)}{\varphi}\right\} \\ &= \frac{1-\lambda\chi}{1-\lambda}\gamma - \xi\frac{\lambda(\chi-1)}{1-\lambda}\frac{\varphi+1}{\varphi}. \end{aligned}$$

Then the stability condition is well-known, and:

$$\frac{\phi_y(1-\beta) + \frac{\psi(\gamma+\varphi)}{\eta}(\phi_p - 1)}{\beta\tilde{\gamma}} > 0.$$

Sufficient conditions are $\tilde{\gamma} > 0$, $\phi_y \geq 0$, and $\phi_p > 1$.

When the stability condition is satisfied, then $\mathbb{E}_t infl_{t+1} = \rho \cdot infl_t$ and $\mathbb{E}_t y_{t+1} = \rho \cdot y_t$, and

$$\begin{aligned} y_t &= \rho \cdot y_t - \frac{int_t - \rho \cdot infl_t}{\tilde{\gamma}} \\ infl_t &= \tilde{\psi} y_t, \\ int_t &= \phi_p \cdot infl_t + \phi_y y_t - m_t, \end{aligned}$$

where

$$\tilde{\psi} := \frac{\psi(\gamma+\varphi)}{\eta(1-\beta\rho)}.$$

Therefore

$$\begin{aligned} y_t &= \rho \cdot y_t - \frac{(\phi_p-\rho)\tilde{\psi}y_t + \phi_y y_t - m_t}{\tilde{\gamma}} \\ &= \frac{m_t}{(1-\rho)\tilde{\gamma} + (\phi_p-\rho)\tilde{\psi} + \phi_y} \\ &= \frac{m_t}{\Omega^{-1} + (\phi_p-\rho)\frac{\psi(\gamma+\varphi)}{\eta(1-\beta\rho)} + \phi_y} < \Omega \end{aligned}$$

where

$$\Omega = \frac{1}{\tilde{\gamma}(1-\rho)} = \frac{1}{\gamma + \frac{\lambda(1-\chi)}{1-\lambda}\left\{\gamma + \xi\frac{(\varphi+1)}{\varphi}\right\}}\frac{1}{1-\rho}.$$

It is proportional to Ω, and also smaller than Ω. The parameter ξ (or κ) gives no impact on other than Ω.

4 Recursive form

An agent j chooses consumption, asset holdings, and hours worked solving standard intertemporal problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t^j, N_t^j)$$

subject to the sequence of the budget constraints:

$$B_t^{N,j} + share_t^j V_t^N \leq Z_t^{N,j} + share_{t-1}^j(V_t^N + D_t^N) + W_t^N N_t^j + P_t TR_t^j - P_t TX_t^j - P_t C_t^j$$

C_t^j,N_t^j are consumption and hours worked, W_t^N is the nominal wage rate of period t, $B_t^{N,j}$ is the nominal value at end of period t of a portfolio of all state-contingent assets held, except for shares in firms, and $X_t^{N,j}$ is the nominal value at beginning of period $t$wealth, except for shares in firms. V_t^N is the nominal market value at time t of shares, and D_t^N their dividend payoff and $share_t^j$ are share holdings.

Absence of arbitrage implies that there exists a stochastic discount factor $Q_{t,t+1}^j$ such that the price at t of a portfolio with uncertain payoff at $t+1$ is as follows:

$$\begin{aligned} B_t^j &= \frac{B_t^{N,j}}{P_t} = E_t Q_{t,t+1}^j \frac{Z_{t+1}^{N,j}}{P_{t+1}} = E_t Q_{t,t+1}^j Z_{t+1}^j \\ V_t &= \frac{V_t^N}{P_t} = E_t Q_{t,t+i}^j \frac{V_{t+1}^N + D_{t+1}^N}{P_{t+1}} = E_t Q_{t,t+i}^j (V_{t+1} + D_{t+1}) \\ &= E_t \sum_{i=0}^{T} Q_{t,t+1}^j D_{t+i} \text{ with } \lim_{i\to\infty} E_t Q_{t,t+i}^j V_{t+i} = 0 \end{aligned}$$

Notice that when $Z_t^{N,j} = INT_{t-1} B_{t-1}^{N,j}$ and $INFL_{t+1} = P_{t+1}/P_t$, then

$$\begin{aligned} B_t^j &= E_t Q_{t,t+1}^j \frac{INT_t B_t^{N,j}}{P_{t+1}} \\ &= E_t Q_{t,t+1}^j \frac{INT_t P_t B_t^j}{P_{t+1}} \\ \Leftrightarrow 1 &= E_t \frac{INT_t}{INFL_{t+1}} Q_{t,t+1}^j, \end{aligned} \tag{10}$$

which is called the Euler equation.

Let $X_t^{N,j}$ (X_t^j) be the nominal (real) value at beginning of period $t$wealth including shares:

$$X_t^{N,j} = Z_t^{N,j} + share_{t-1}^j(V_t^N + D_t^N)$$
$$X_t^j = \frac{X_t^{N,j}}{P_t} = Z_t^j + share_{t-1}^j E_t \sum_{i=0}^{T} Q_{t,t+1}^j D_{t+i}.$$

Then

$$B_t^j + share_t^j V_t = E_t Q_{t,t+1}^j X_{t+1}^j.$$

Therefore, the budget constraints in real term implies:

$$\begin{aligned}
X_t^j &\geq C_t^j + TX_t^j - TR_t^j - W_t N_t^j + E_t Q_{t,t+1}^j X_{t+1}^j \\
&\geq C_t^j + TX_t^j - TR_t^j - W_t N_t^j + E_t Q_{t,t+1}^j (C_{t+1}^j + TX_{t+1}^j - TR_{t+1}^j - W_{t+1} N_{t+1}^j + Q_{t+1,t+2}^j X_{t+2}^j) \\
&= C_t^j + TX_t^j - TR_t^j - W_t N_t^j + E_t Q_{t,t+1}^j (C_{t+1}^j + TX_{t+1}^j - TR_{t+1}^j - W_{t+1} N_{t+1}^j) + E_t Q_{t,t+2}^j X_{t+2}^j \\
&\geq E_t \sum_{i=0}^{T} Q_{t,t+i}^j \left\{ C_{t+i} + TX_{t+i}^j - TR_{t+i}^j - W_{t+i} N_{t+i} \right\} + E_t Q_{t,T+1}^j X_{T+1}^j \\
&\geq E_t \sum_{i=0}^{\infty} Q_{t,t+i}^j \left\{ C_{t+i} + TX_{t+i}^j - TR_{t+i}^j - W_{t+i} N_{t+i} \right\} \text{ with } \lim_{T\to\infty} E_t Q_{t,T+1}^j X_{T+1}^j = 0.
\end{aligned}$$

Thus, we obtain the intertemporal budget constraints:

$$\begin{aligned}
E_t \sum_{i=0}^{\infty} Q_{t,t+i}^j C_{t+i} &\leq X_t^j + E_t \sum_{i=0}^{T} Q_{t,t+i}^j \left\{ W_{t+i} N_{t+i} + TR_{t+i}^j - TX_{t+i}^j \right\} \\
&= Z_t^j + E_t \sum_{i=0}^{T} Q_{t,t+i}^j \left\{ W_{t+i} N_{t+i} + TR_{t+i}^j - TX_{t+i}^j + share_{t-1}^j D_{t+i} \right\} \\
&= Z_t^j + E_t \sum_{i=0}^{\infty} Q_{t,t+i}^j Y_{t+i}^{D,j} \qquad (11)
\end{aligned}$$

where $Y_t^{D,j}$ is the real disposable income.

The first-order conditions at each date and each state:

$$Q_{t,t+1}^j = \beta \frac{U_C(C_{t+1}^j)}{U_C(C_t^j)}$$

along with the intertemporal budget constraints (11) with equality and transversality conditions: $\lim_{i\to\infty} E_t Q_{t,t+i}^j Z_{t+i}^j = \lim_{i\to\infty} E_t Q_{t,t+i}^j V_{t+i} = 0$.

In our model with equilibrium, $Z_t^j = (INT_{t-1}/INFL_t)B_{t-1}^j = 0$ and $share_{t-1}^j = 1$. Therefore,

$$\sum_{i=0}^{\infty} \beta^i \frac{U_C(C_{t+i}^j)}{U_C(C_t^j)} C_{t+i}^j = \sum_{i=0}^{\infty} \beta^i \frac{U_C(C_{t+i}^j)}{U_C(C_t^j)} Y_{t+i}^{D,j}.$$

Denote by small letter log deviations from steadystate, except for rates of return (where they denote absolute deviations). Notice that

$$Q_{t,t+i}^j = \beta^i \frac{U_C(C_{t+i}^j)}{U_C(C_t^j)}$$

and in steady state: $Q_i^j = Q_i = \beta^i$. Notice the Euler equation implies:

$$int_t - E_t infl_{t+1} + E_t q_{t+1}^j = 0.$$

Thus we have

$$q_{t,t+i}^j = \ln \frac{Q_{t,t+i}^j}{Q_{t,t}^j} = \ln \frac{U_C(C_{t+i}^j)}{U_C(C_t^j)} = -(\gamma+\kappa)(c_{t+i}^j - c_t^j) + \kappa(n_{t+i}^j - n_t^j)$$

or

$$(\gamma+\kappa)c_t^j - \kappa n_t^j = (\gamma+\kappa)E_t c_{t+i}^j - \kappa E_t n_{t+i}^j + E_t q_{t,t+i}^j. \tag{12}$$

Using the real interest rate

$$r_t := int_t - E_t infl_{t+1} = -E_t q_{t,t+1}^j,$$

we can rewrite the above as follows:

$$(\gamma+\kappa)c_t^j - \kappa n_t^j = (\gamma+\kappa)E_t c_{t+1}^j - \kappa E_t n_{t+1}^j - r_t.$$

Since

$$\begin{aligned}
(\gamma+\kappa)E_t c_{t+1}^j - \kappa E_t n_{t+1}^j &= (\gamma+\kappa)E_t c_{t+2}^j - \kappa E_t n_{t+2}^j - E_t r_{t+1}, \\
&\vdots \\
(\gamma+\kappa)E_t c_{t+i-1}^j - \kappa E_t n_{t+i-1}^j &= (\gamma+\kappa)E_t c_{t+i}^j - \kappa E_t n_{t+i}^j - E_t r_{t+i-1},
\end{aligned}$$

we obtain

$$(\gamma+\kappa)c_t^j - \kappa n_t^j = (\gamma+\kappa)E_t c_{t+i}^j - \kappa E_t n_{t+i}^j - E_t \sum_{k=0}^{i-1} r_{t+k}.$$

Because of (12), we get

$$q_{t,t+i}^j = -\sum_{k=0}^{i-1} r_{t+k}. \tag{13}$$

Now loglinearize intertemporal budget constraint

$$E_t \sum_{i=0}^{\infty} \beta^i (q_{t,t+i}^j + c_{t+i}^j) = E_t \sum_{i=0}^{\infty} \beta^i (q_{t,t+i}^j + y_{t+i}^{D,j}).$$

Multiplying to each side $(\gamma+\kappa)$, we obtain:

$$(\gamma+\kappa)E_t \sum_{i=0}^{\infty} \beta^i (q_{t,t+i}^j + c_{t+i}^j) = (\gamma+\kappa)E_t \sum_{i=0}^{\infty} \beta^i (q_{t,t+i}^j + y_{t+i}^{D,j}).$$

Then adding to each side $(1-\kappa-\gamma)E_t \sum_{i=0}^{\infty} \beta^i q_{t,t+i}^j - \gamma E_t \sum_{i=0}^{\infty} \beta^i n_{t,t+i}^j$, we obtain

$$E_t \sum_{i=0}^{\infty} \beta^i ((\gamma+\kappa)c_{t+i}^j - \kappa n_{t+i}^j + q_{t,t+i}^j) = E_t \sum_{i=0}^{\infty} \beta^i ((\gamma+\kappa)y_{t+i}^{D,j} - \kappa n_{t+i}^j + q_{t,t+i}^j)$$

By virtue of the Euler equation (12) the LHS simplifies

$$E_t \sum_{i=0}^{\infty} \beta^i ((\gamma+\kappa)c_{t+i}^j - \kappa n_{t+i}^j + q_{t,t+i}^j) = \{(\gamma+\kappa)c_t^j - \kappa n_t^j\} \sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}\{(\gamma+\kappa)c_t^j - \kappa n_t^j\}$$

From RHS, using (13) and

$$
\begin{aligned}
\sum_{i=0}^{\infty}\beta^i q_{t,t+i}^j &= -\sum_{i=0}^{\infty}\beta^i\sum_{k=0}^{i-1}r_{t+k} \\
&= -\beta r_t - \beta^2(r_t + r_{t+1}) + \cdots + \\
&- \beta^i(r_t + r_{t+1} + \cdots + r_{t+i-1}) + \cdots \\
&= -\beta\frac{r_t}{1-\beta} - \beta^2\frac{r_{t+1}}{1-\beta} - \cdots - -\beta^i\frac{r_{t+i-1}}{1-\beta} - \cdots \\
&= -\frac{\beta}{1-\beta}\sum_{i=0}^{\infty}\beta^i r_{t+i},
\end{aligned}
$$

and multiplying by $1-\beta$, we obtain

$$
\begin{aligned}
(\gamma+\kappa)c_t^j - \kappa n_t^j &= -\beta\sum_{i=0}^{\infty}\beta^i E_t r_{t+i} + (\gamma+\kappa)(1-\beta)E_t\sum_{i=0}^{\infty}\beta^i y_{t+i}^{D,j} - \kappa(1-\beta)E_t\sum_{i=0}^{\infty}\beta^i n_{t+i}^j \\
&= -\beta r_t + (\gamma+\kappa)(1-\beta)y_t^{D,j} - \kappa(1-\beta)n_t^j - \beta\sum_{i=1}^{\infty}\beta^i E_t r_{t+i} \\
&+ (\gamma+\kappa)(1-\beta)E_t\sum_{i=0}^{\infty}\beta^i y_{t+i}^{D,j} - \kappa(1-\beta)E_t\sum_{i=1}^{\infty}\beta^i n_{t+i}^j \\
&= -\beta r_t + (\gamma+\kappa)(1-\beta)y_t^{D,j} - \kappa(1-\beta)n_t^j - \beta\sum_{i=0}^{\infty}\beta^{i+1}E_t r_{t+i+1} \\
&+ (\gamma+\kappa)(1-\beta)E_t\sum_{i=0}^{\infty}\beta^{i+1}y_{t+i+1}^{D,j} - \kappa(1-\beta)E_t\sum_{i=1}^{\infty}\beta^i n_{t+i}^j \\
&= -\beta r_t + (\gamma+\kappa)(1-\beta)y_t^{D,j} - \kappa(1-\beta)n_t^j + \beta(\gamma+\kappa)E_t c_{t+1}^j - \beta\kappa E_t n_{t+1}^j
\end{aligned}
$$

In sum,

$$
\begin{aligned}
(\gamma+\kappa)c_t^j &= -\beta r_t + (\gamma+\kappa)(1-\beta)y_t^{D,j} + \beta\kappa n_t^j + \beta(\gamma+\kappa)E_t c_{t+1}^j - \beta\kappa E_t n_{t+1}^j \\
\Leftrightarrow c_t^j &= \beta E_t c_{t+1}^j + (1-\beta)y_t^{D,j} - \frac{\beta}{\gamma+\kappa}r_t + \frac{\beta\kappa}{\gamma+\kappa}n_t^j - \frac{\beta\kappa}{\gamma+\kappa}E_t n_{t+1}^j \\
&= (1-\beta)y_t^{D,j} + \beta\left(\mathbb{E}_t c_{t+1}^j - \frac{1}{\gamma+\kappa}\left(r_t + \kappa\left(\mathbb{E}_t n_{t+1}^j - n_t^j\right)\right)\right).
\end{aligned}
$$

5 NK consumption function with non-separable TANK model

- TANK model:

$$
\begin{aligned}
c_t^H &= \chi y_t \\
c_t^S &= \frac{1-\lambda\chi}{1-\lambda} y_t \\
n_t^H &= (1 + (1-\chi)\sigma/\varphi) y_t \\
n_t^S &= (1 - \frac{\lambda(1-\chi)}{1-\lambda}\sigma/\varphi) y_t \\
&= \left(\frac{1-\lambda\chi}{1-\lambda} + \frac{\lambda(\chi-1)}{1-\lambda} - \frac{\lambda(1-\chi)}{1-\lambda}\frac{\gamma}{\varphi} \right) y_t \\
&= \left(\frac{1-\lambda\chi}{1-\lambda} - \frac{\lambda(1-\chi)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right) y_t \\
&= \left(\frac{1-\lambda\chi}{1-\lambda} + \frac{\lambda(\chi-1)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right) y_t
\end{aligned}
$$

- Type S consumption function:

$$
\begin{aligned}
c_t^S &= \beta E_t c_{t+1}^S + (1-\beta) y_t^S - \frac{\beta r_t}{\gamma+\kappa} + \frac{\beta\kappa}{\gamma+\kappa}(n_t^S - E_t n_{t+1}^S) \\
&= \frac{1-\lambda\chi}{1-\lambda}\{\beta E_t c_{t+1} + (1-\beta) y_t\} - \frac{\beta r_t}{\gamma+\kappa} \\
&+ \frac{\beta\kappa}{\gamma+\kappa}\left(\frac{1-\lambda\chi}{1-\lambda} + \frac{\lambda(\chi-1)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right)(y_t - E_t c_{t+1}) \\
&= \left\{ \beta\frac{1-\lambda\chi}{1-\lambda} - \frac{\kappa\beta}{\gamma+\kappa}\left(\frac{1-\lambda\chi}{1-\lambda} + \frac{\lambda(\chi-1)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right) \right\} E_t c_{t+1} \\
&+ \left\{ (1-\beta)\frac{1-\lambda\chi}{1-\lambda} + \frac{\kappa\beta}{\gamma+\kappa}\left(\frac{1-\lambda\chi}{1-\lambda} + \frac{\lambda(\chi-1)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right) \right\} y_t - \frac{\beta r_t}{\gamma+\kappa} \\
&= \left\{ \frac{1-\lambda\chi}{1-\lambda}\tilde{\beta} - \frac{\kappa\beta}{\gamma+\kappa}\frac{\lambda(\chi-1)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right\} E_t c_{t+1} \\
&+ \left\{ \frac{1-\lambda\chi}{1-\lambda}(1-\tilde{\beta}) + \frac{\kappa\beta}{\gamma+\kappa}\frac{\lambda(\chi-1)}{1-\lambda}\frac{\gamma+\varphi}{\varphi} \right\} y_t - \frac{\tilde{\beta} r_t}{\gamma} \\
&= \left\{ \frac{1-\lambda\chi}{1-\lambda}\tilde{\beta} + \kappa\tilde{\beta}\frac{\lambda(1-\chi)}{1-\lambda}\left(\frac{1}{\varphi} + \frac{1}{\gamma} \right) \right\} E_t c_{t+1} \\
&+ \left\{ \frac{1-\lambda\chi}{1-\lambda}(1-\tilde{\beta}) - \kappa\tilde{\beta}\frac{\lambda(1-\chi)}{1-\lambda}\left(\frac{1}{\varphi} + \frac{1}{\gamma} \right) \right\} y_t - \frac{\tilde{\beta} r_t}{\gamma}
\end{aligned}
$$

- Aggregate dynamic consumption function:

$$
\begin{aligned}
c_t &= \lambda c_t^S + (1-\lambda)c_t^H \\
&= \lambda\chi y_t + \left\{(1-\lambda\chi)\tilde{\beta} + \kappa\tilde{\beta}\lambda(1-\chi)\left(\frac{1}{\varphi}+\frac{1}{\gamma}\right)\right\} E_t c_{t+1} \\
&+ \left\{(1-\lambda\chi)(1-\tilde{\beta}) - \kappa\tilde{\beta}\lambda(1-\chi)\left(\frac{1}{\varphi}+\frac{1}{\gamma}\right)\right\} y_t - \frac{(1-\lambda)\tilde{\beta}r_t}{\gamma} \\
&= \left\{(1-\lambda\chi)\tilde{\beta} + \kappa\tilde{\beta}\lambda(1-\chi)\left(\frac{1}{\varphi}+\frac{1}{\gamma}\right)\right\} E_t c_{t+1} \\
&+ \left\{\lambda\chi + (1-\lambda\chi)(1-\tilde{\beta}) - \kappa\tilde{\beta}\lambda(1-\chi)\left(\frac{1}{\varphi}+\frac{1}{\gamma}\right)\right\} y_t - \frac{(1-\lambda)\tilde{\beta}r_t}{\gamma} \\
&= \tilde{\beta}\left\{1-\lambda\chi + \kappa\lambda(1-\chi)\left(\frac{1}{\varphi}+\frac{1}{\gamma}\right)\right\} E_t c_{t+1} \\
&+ \left\{1-\tilde{\beta}\left(1-\lambda\chi + \kappa\lambda(1-\chi)\left(\frac{1}{\varphi}+\frac{1}{\gamma}\right)\right)\right\} y_t - \frac{(1-\lambda)\tilde{\beta}r_t}{\gamma} \\
&= \frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma} E_t c_{t+1} + \left\{1 - \frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma}\right\} y_t - \frac{(1-\lambda)\tilde{\beta}r_t}{\gamma} \\
\tilde{\gamma} &= \frac{1-\lambda\chi}{1-\lambda}\gamma - \xi\frac{\lambda(\chi-1)}{1-\lambda}\frac{\varphi+1}{\varphi}
\end{aligned}
$$

- Aggregate dynamic consumption function:

$$
\begin{aligned}
c_t &= \frac{1-\frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma}}{1-\frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma}\rho} y_t - \frac{\frac{(1-\lambda)\tilde{\beta}}{\gamma}r_t}{1-\frac{(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma}\rho} \\
&= \frac{\gamma-(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma-(1-\lambda)\tilde{\beta}\tilde{\gamma}\rho} y_t - \frac{(1-\lambda)\tilde{\beta}}{\gamma-(1-\lambda)\tilde{\beta}\tilde{\gamma}\rho} r_t
\end{aligned}
$$

- MPC:

$$
\begin{aligned}
\omega &= \frac{\gamma-(1-\lambda)\tilde{\beta}\tilde{\gamma}}{\gamma-(1-\lambda)\tilde{\beta}\tilde{\gamma}\rho} \\
&= \frac{\gamma-(1-\lambda)\tilde{\beta}\left\{\frac{1-\lambda\chi}{1-\lambda}\gamma - \xi\frac{\lambda(\chi-1)}{1-\lambda}\frac{\varphi+1}{\varphi}\right\}}{\gamma-(1-\lambda)\tilde{\beta}\tilde{\gamma}\rho\left\{\frac{1-\lambda\chi}{1-\lambda}\gamma - \xi\frac{\lambda(\chi-1)}{1-\lambda}\frac{\varphi+1}{\varphi}\right\}} \\
&= \frac{\gamma-\tilde{\beta}\left\{(1-\lambda\chi)\gamma - \xi\lambda(\chi-1)\frac{\varphi+1}{\varphi}\right\}}{\gamma-\tilde{\beta}\rho\left\{(1-\lambda\chi)\gamma - \xi\lambda(\chi-1)\frac{\varphi+1}{\varphi}\right\}} \\
&= \frac{1-\tilde{\beta}\left\{1-\lambda\chi - \xi\lambda(\chi-1)\frac{\varphi+1}{\gamma\varphi}\right\}}{1-\tilde{\beta}\rho\left\{1-\lambda\chi - \xi\lambda(\chi-1)\frac{\varphi+1}{\gamma\varphi}\right\}}
\end{aligned}
$$

6 Proof of Proposition 3

- Note

$$\frac{X_t - X}{X} \approx x_t + \frac{1}{2}x_t^2$$
$$\left(\frac{X_t - X}{X}\right)^2 \approx x_t^2$$

- Second-Order Approximation to Utility function:

$$\begin{aligned}
U(C_t^j, N_t^j) \approx\ & U(C_{SS}^j, N^j) \\
& + U_C C_{SS}^j \frac{C_t^j - C^j}{C^j} + \frac{U_{CC}}{2}(C^j)^2 \left(\frac{C_t^j - C^j}{C^j}\right)^2 \\
& + U_N N^j \frac{N_t^j - N^j}{N^j} + \frac{U_{NN}}{2}(N^j)^2 \left(\frac{N_t^j - N^j}{N^j}\right)^2 \\
& + U_{CN} C^j N^j \frac{C_t^j - C^j}{C^j} \frac{N_t^j - N^j}{N^j} \\
& + U_C C^j (c_t^j + \frac{1}{2}(c_t^j)^2) + \frac{U_{CC}^j}{2}(C_{SS}^j)^2 (c_t^j)^2 \\
& + U_N N^j (n_t^j + \frac{1}{2}(n_t^j)^2) + \frac{U_{NN}^j}{2}(N^j)^2 (n_t^j)^2 \\
& + U_{CN} C^j N^j c_t^j n_t^j
\end{aligned}$$

- Thus

$$\begin{aligned}
\frac{U_t^j - U^j}{U_C^j C^j} =\ & c_t^j + \frac{1}{2}(c_t^j)^2 + \frac{U_{CC}^j C^j}{2U_C^j}(c_t^j)^2 \\
& + \frac{N^j}{C^j}\frac{U_N^j}{U_C^j}\left(n_t^j + \frac{1}{2}(n_t^j)^2 + \frac{U_{NN}^j N^j}{2U_N^j}(n_t^j)^2\right) \\
& + \frac{U_{CN}^j N^j}{U_C^j} c_t^j n_t^j.
\end{aligned}$$

- Using

$$\begin{aligned}
\kappa^j &= -U_{CN}^j C^j / U_N^j, \\
\gamma &= -\frac{U_{CC}^j C^j}{2U_C^j} + \frac{U_{CN}^j C^j}{U_N^j} = -\frac{U_{CC}^j C^j}{2U_C^j} - \kappa^j, \\
\varphi &= \frac{U_{NN}^j N^j}{U_N^j} - \frac{U_{CN}^j N^j}{U_C^j} = \frac{U_{NN}^j N^j}{U_N^j} + \frac{N^j}{C^j}\frac{U_N^j}{U_C^j}\kappa^j,
\end{aligned}$$

we obtain

$$
\begin{aligned}
\frac{U^j_t - U^j}{U^j_C C^j} &= c^j_t + \frac{1}{2}(c^j_t)^2 + \frac{-\gamma - \kappa^j}{2}(c^j_t)^2 \\
&+ \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\left(n^j_t + \frac{1}{2}(n^j_t)^2 + \frac{\varphi - \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\kappa^j}{2}(n^j_t)^2\right) \\
&- \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\kappa^j c^j_t n^j_t \\
&= c^j_t + \frac{1-\gamma}{2}(c^j_t)^2 + \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\left(n^j_t + \frac{1+\varphi}{2}(n^j_t)^2\right) \\
&- \frac{\kappa^j}{2}(c^j_t)^2 - \left(\frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\right)^2(n^j_t)^2 - \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\kappa^j c^j_t n^j_t \\
&= c^j_t + \frac{1-\gamma}{2}(c^j_t)^2 + \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}\left(n^j_t + \frac{1+\varphi}{2}(n^j_t)^2\right) \\
&- \frac{\kappa^j}{2}\left(c^j_t + \frac{N^j}{C^j}\frac{U^j_N}{U^j_C}n^j_t\right)^2.
\end{aligned}
$$

- When $\frac{N^j}{C^j}\frac{U^j_N}{U^j_C} = \frac{N}{C}\frac{U_N}{U_C} = 1$ and $\kappa^j = \kappa$, then

$$
\frac{U^j_t - U^j}{U^j_C C^j} = c^j_t + \frac{1-\gamma}{2}(c^j_t)^2 - \left(n^j_t + \frac{1+\varphi}{2}(n^j_t)^2\right) - \frac{1}{2}\kappa\left(c^j_t - n^j_t\right)^2.
$$

- The objective function under TANK $\mathbb{W}$ is as follows:

$$
\begin{aligned}
\mathbb{W} &= \lambda\frac{U^H_t - U^H}{U^H_C C^H} + (1-\lambda)\frac{U^S_t - U^S}{U^S_C C^S} \\
&= \lambda c^H_t + (1-\lambda)c^S_t + \frac{1-\gamma}{2}\{\lambda(c^H)^2 + (1-\lambda)(c^S)^2\} \\
&- \lambda n^H_t + (1-\lambda)n^S_t - \frac{1+\varphi}{2}\{\lambda(n^H_t)^2 + (1-\lambda)(n^S_t)^2\} \\
&- \frac{\kappa}{2}\{\lambda(c^H - n^H_t)^2 + (1-\lambda)\left(c^S_t - n^S_t\right)^2\}.
\end{aligned}
$$

- When we use

$$
\begin{aligned}
c_t &= \lambda c^H_t + (1-\lambda)c^S_t + \frac{1}{2}\{\lambda(c^H)^2 + (1-\lambda)(c^S)^2\}, \\
n_t &= \lambda n^H_t + (1-\lambda)n^S_t + \frac{1}{2}\{\lambda(n^H)^2 + (1-\lambda)(n^S)^2\},
\end{aligned}
$$

then we obtain

$$
\begin{aligned}
\mathbb{W} &= c_t - n_t - \frac{\gamma}{2}\{\lambda(c^H)^2 + (1-\lambda)(c^S)^2\} \\
&- \frac{\varphi}{2}\{\lambda(n^H_t)^2 + (1-\lambda)(n^S_t)^2\} \\
&- \frac{\kappa}{2}\{\lambda(c^H - n^H_t)^2 + (1-\lambda)\left(c^S_t - n^S_t\right)^2\}
\end{aligned}
$$

- Since

$$
\begin{aligned}
y_t &= c_t + \frac{\eta}{2} infl_t^2, \\
y_t &= n_t,
\end{aligned}
$$

then

$$
\begin{aligned}
\mathbb{W} = &-\frac{\eta}{2} infl_t^2 - \frac{\gamma}{2}\{\lambda(c^H)^2 + (1-\lambda)(c^S)^2\} \\
&- \frac{\varphi}{2}\{\lambda(n_t^H)^2 + (1-\lambda)(n_t^S)^2\} \\
&- \frac{\kappa}{2}\{\lambda(c^H - n_t^H)^2 + (1-\lambda)\left(c_t^S - n_t^S\right)^2\}.
\end{aligned}
$$

- Since

$$
\begin{aligned}
c_t^H &= \chi y_t, \\
c_t^S &= \frac{1-\lambda\chi}{1-\lambda} y_t,
\end{aligned}
$$

then

$$
\begin{aligned}
\lambda(c^H)^2 + (1-\lambda)(c^S)^2 &= \lambda\left(\chi y_t\right)^2 + (1-\lambda)\left(\frac{1-\lambda\chi}{1-\lambda} y_t\right)^2 \\
&= \lambda\chi^2 y_t^2 + \frac{(1-\lambda\chi)^2}{1-\lambda} y_t^2 \\
&= \left(1 + \frac{\lambda(\chi-1)^2}{1-\lambda}\right) y_t^2.
\end{aligned}
$$

- Since

$$
\begin{aligned}
n_t^H &= \left(1 + \frac{\gamma}{\varphi}(1-\chi)\right) y_t, \\
n_t^S &= \left(1 + \frac{\gamma}{\varphi}\frac{\lambda}{1-\lambda}(\chi-1)\right) y_t,
\end{aligned}
$$

then

$$
\begin{aligned}
&\lambda(n_t^H)^2 + (1-\lambda)(n_t^S)^2 \\
&= \lambda\left(1 + \frac{\sigma}{\varphi}(1-\chi)\right)^2 y_t^2 + (1-\lambda)\left(1 + \frac{\sigma}{\varphi}\frac{\lambda}{1-\lambda}(\chi-1)\right)^2 y_t^2 \\
&= \left(1 + \lambda\left(\frac{\sigma}{\varphi}\right)^2 (1-\chi)^2 + \frac{\lambda^2}{1-\lambda}\left(\frac{\sigma}{\varphi}\right)^2 (1-\chi)^2\right) y_t^2 \\
&= \left(1 + \frac{\lambda}{1-\lambda}\left(\frac{\sigma}{\varphi}\right)^2 (1-\chi)^2\right) y_t^2.
\end{aligned}
$$

- Since

$$
\begin{aligned}
c_t^H - n_t^H &= \left(\frac{\sigma}{\varphi} + 1\right)(\chi-1) y_t, \\
c_t^S - n_t^S &= \left(\frac{1-\lambda\chi}{1-\lambda} - 1 - \frac{\gamma}{\varphi}\frac{\lambda}{1-\lambda}(\chi-1)\right) y_t \\
&= \left(\frac{-\lambda(\chi-1)}{1-\lambda} - \frac{\gamma}{\varphi}\frac{\lambda}{1-\lambda}(\chi-1)\right) y_t \\
&= -\frac{\lambda}{1-\lambda}\left(1 + \frac{\gamma}{\varphi}\right)(\chi-1) y_t,
\end{aligned}
$$

then

$$
\begin{aligned}
&\lambda(c^H - n_t^H)^2 + (1-\lambda)\left(c_t^S - n_t^S\right)^2 \\
&= \lambda\left(\frac{\gamma}{\varphi}+1\right)^2(\chi-1)^2 y_t^2 \\
&+ \frac{\lambda^2}{1-\lambda}\left(\frac{\gamma}{\varphi}+1\right)^2(\chi-1)^2 y_t^2 \\
&= \frac{\lambda}{1-\lambda}\left(\frac{\gamma}{\varphi}+1\right)^2(\chi-1)^2 y_t^2.
\end{aligned}
$$

- Thus

$$
\begin{aligned}
\mathbb{W} &= -\eta\frac{infl^2}{2} - \frac{\gamma}{2}\left(1+\frac{\lambda(\chi-1)^2}{1-\lambda}\right)y_t^2 \\
&- \frac{\varphi}{2}\left(1+\frac{\lambda}{1-\lambda}\left(\frac{\gamma}{\varphi}\right)^2(1-\chi)^2\right)y_t^2 \\
&- \kappa\frac{\lambda}{1-\lambda}\left(\frac{\gamma}{\varphi}+1\right)^2(\chi-1)^2 y_t^2 \\
&= -\eta\frac{infl^2}{2} - \frac{1}{2}\left(\gamma+\gamma\frac{\lambda(\chi-1)^2}{1-\lambda}+\varphi+\varphi\frac{\lambda}{1-\lambda}\left(\frac{\gamma}{\varphi}\right)^2(1-\chi)^2\right)y_t^2 \\
&- \kappa\frac{\lambda}{1-\lambda}\left(\frac{\gamma}{\varphi}+1\right)^2(\chi-1)^2 y_t^2 \\
&= -\eta\frac{infl^2}{2} - \frac{1}{2}\left(\gamma+\varphi+\gamma\frac{\gamma+\varphi}{\varphi}\frac{\lambda(\chi-1)^2}{1-\lambda}\right)y_t^2 \\
&- \xi\frac{\varphi+1}{\varphi}\frac{\gamma+\varphi}{\varphi}\lambda(1-\lambda)\left(\frac{\chi-1}{1-\lambda}\right)^2 y_t^2.
\end{aligned}
$$

- When we use

$$
gap_t = \frac{1-\chi}{1-\lambda}y_t,
$$

then we obtain:

$$
\begin{aligned}
\mathbb{W} &= -\eta\frac{infl_t^2}{2} - \frac{\gamma+\varphi}{2}y_t^2 \\
&- \frac{\gamma+\varphi}{2}\frac{\gamma}{\varphi}\lambda(\chi-1)gap_t^2 \\
&- \frac{\gamma+\varphi}{2}\frac{\xi}{\varphi}\frac{\varphi+1}{\varphi}\lambda(1-\lambda)gap_t^2 \\
&= -\frac{\eta}{2}infl_t^2 - \frac{\gamma+\varphi}{2}\left(y_t^2+\left\{\frac{\gamma}{\varphi}+\frac{\xi}{\varphi}\frac{\varphi+1}{\varphi}\right\}\lambda(1-\lambda)gap_t^2\right).
\end{aligned}
$$