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Yutaka Suzuki

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by

Yutaka Suzuki

Faculty of Economics, Hosei University

4342 Aihara, Machida-City, Tokyo 194-0298, Japan

E-Mail: yutaka@mt.tama.hosei.ac.jp

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Abstract

We apply the monotonicity properties of optimal solutions, due to so-called "single crossing property", to a context of nonstationary dynamic games and obtain "early turnpike results". First, in a setting of dynamic games of common pool resource, with the powerful results by recent "Monotone comparative statics" literature in hand, we prove the existence of a "forecast horizon", which indicates that after all, "backward induction" can be performed for infinite horizon games, in the sense that, in order to solve for the first period equilibrium, one may consider the finite horizon dynamic game whose planning horizon is exactly the forecast horizon. Then, we investigate more foundation on "monotonicity of optimal solutions (equilibrium outcomes)" by extensively using "Monotone Comparative Statics", and more generalization of under what conditions our theorems remain to hold. We also consider some interesting economic applications, and last present a "robust prediction" on the condition for the existence of "Early Turnpike" and "Forecast Horizon" in dynamic games.

***Key words*:** Monotone Comparative Statics, Nonstationary Dynamic Games, Markov Perfect Equilibria, Forecast Horizon, Time-varying Renewal Dynamics

***JEL Classification*:** C62, C73, L13

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1. Introduction

We apply the monotonicity properties of optimal solutions to dynamic optimization problems to obtain "early turnpike results", in the context of dynamic games. For the sake of clarity, we will focus our attention in a specific setting, that of games of exploitation of a common pool resource. These games are the most used model for competitive fisheries, extraction of oil resource, and, the problem of commons ala Hume (1739) and Hardin (1968). In this context, *monotonicity properties* are also a pervasive feature: the higher the available stock, the higher the consumption levels in equilibrium. Of course, we can also consider other aspects of *monotonicity properties*: the higher the renewal of stock, or the longer the time horizon, how will the consumption levels change in equilibrium? Recently, a lot of research efforts have been oriented towards the analysis of monotonic behavior of optimal solutions and equilibrium outcomes, also referred to as "Monotone Comparative Statics" (see Milgrom and Roberts (1994), Milgrom and Shannon (1994), Villas-Boas (1997), Topkis (1998), Edlin and Shannon (1998), and Athey (2001)). With these powerful results in hand, we prove the existence of a "forecast horizon" in a dynamic game. Interestingly enough, the existence of such horizon indicates that after all, "backward induction" can be performed for infinite horizon games, in the sense that, to solve for the first period equilibrium, one may consider the finite horizon dynamic game whose planning horizon is exactly the forecast horizon. Additionally, this result can also be used to argue that the concept of *subgame perfect* equilibrium does not require "hyper rational" behavior by the players engaged in an infinite horizon game, since the first stage game gets *decoupled* from the tail effects.

In this paper, we investigate more foundation on "monotonicity of optimal solutions (equilibrium outcomes)" by using "Monotone Comparative Statics" extensively, and more generalization of under what conditions our theorems remain to hold, and does not, e.g., whether strategic relatedness (strategic substitutability or complementality) does matter or not, and super-modularity/sub-modularity in what variables does matter. We examine some interesting economic applications, and present a "robust prediction" for the condition on the existence of "Early Turnpike" and "Forecast Horizon".

2. Dynamic Games of 'Common Pool Resource'

2.1 Set up

We present the class of nonstationary dynamic games of extraction of common pool resources (CPR). Given "extraction "levels e_t^1 and e_t^2 by players 1 and 2 at time period t, and the current level x_t of the Common Property resource, the dynamics (the evolution of the state variable) is modeled by:

$$x_{t+1} = f_t\left(x_t - e_t^1 - e_t^2\right) \quad t = 0,1,2,\ldots$$

where we assume that $f_t(0) = 0$ and it is monotone increasing. Moreover, there exists $\bar{x}_t > 0$ such that for every $x > \bar{x}_t$, we have:

$$f_t(x) < x$$

Hence, the state space at time period $t+1$ is $S_{t+1} \in [0, x_t]$, regardless of "extraction "levels e_t^1 and e_t^2. Moreover, let us assume that:

$$M = \sup_t \bar{x}_t < \infty$$

and we will denote by $S = [0, M]$, the set of feasible states. Players have bounded maximum "extraction" levels[1], i.e.:

$$0 \le e_t^i \le K^i(x_t) \subseteq \left[0, \frac{x_t}{2}\right] \qquad i = 1,2$$

The payoffs for the Infinite Horizon Game are:

$$\sum_{t=0}^{\infty} \delta^t \cdot U_t^i\left(e_t^1, e_t^2\right) \qquad i = 1,2$$

where $\delta \in [0,1)$ is a common discount factor, and we assume that:

$$\sup_{i,t,e} U_t^i\left(e_t^1, e_t^2\right) \le M < \infty$$

Under these basic assumptions, this class of dynamic games is "*continuous at infinity*" as defined in Fudenberg and Levine (1983)

2.2 Finite Horizon Markov Perfect Equilibria.

A strategy for player i is a sequence of maps of the form:

$$\gamma_i^T = \left(\gamma_0^i, \gamma_1^i, \gamma_2^i, \ldots \gamma_{T-1}^i\right)$$

where $\gamma_t^i : S_t \mapsto R^+$.[2] We will restrict our attention to ***"feasible" strategies***, i.e. $i = 1,2$:

$$\gamma_t^i(x_t) \in \left[0, K^i(x_t)\right] \qquad t = 0,1,\ldots,T-1$$

[1] Note that under this assumption, there is never overexploitation.

[2] This is a *Markov* strategy, in that strategies depend only on specified state variables S_t. See, Fudenberg and Tirole (1991).

Definition 1 *A pair of strategies* $\bar{\gamma}^T = \left(\bar{\gamma}_1^T, \bar{\gamma}_2^T\right)$ *is called a* ***Nash Equilibrium of the dynamic game*** *if for every feasible pair* $\left(\gamma_1^T, \gamma_2^T\right)$*:*

$$\sum_{t=0}^{T}\delta^t\, U_t^1\left(\bar{\gamma}_t^1(x_t), \bar{\gamma}_t^2(x_t)\right) \geq \sum_{t=0}^{T}\delta^t \cdot U_t^1\left(\gamma_t^1(x_t), \bar{\gamma}_t^2(x_t)\right)$$
$$\sum_{t=0}^{T}\delta^t\, U_t^2\left(\bar{\gamma}_t^1(x_t), \bar{\gamma}_t^2(x_t)\right) \geq \sum_{t=0}^{T}\delta^t \cdot U_t^2\left(\bar{\gamma}_t^1(x_t), \gamma_t^2(x_t)\right)$$

In words, $\bar{\gamma}^T$ is said to be *a Nash equilibrium* if and only if for every player $i = 1,2$ who would like to deviate from $\bar{\gamma}^T$ by playing another feasible strategy $\gamma^{T'}$, he or she would have *no incentive to deviate* as far as the other player follows $\bar{\gamma}^T$. More specifically, this says that no player gains strictly more by following a different extraction plan from the initial state x_0.

This solution concept is known not to be **time consistent** in the sense that the play prescribed after some state other than initial state may not constitute itself Nash equilibrium for the game that starts at such state. In other words, the play off the equilibrium history is not *"credible"*, thereby, implicit "threat" of such off equilibrium play may not be taken seriously by the other player. To rule out such "non-credible threat" for the deviator, a refinement of the previous equilibrium concept is to require that the play to follow after any other state $x_k \in S_k$ at time period $0 \leq k < T$ must prescribe a Nash equilibrium in the sense above defined for the game that starts at x_k, which is known most commonly as a *subgame.*[3]

Now, as a new concept to avoid *non-credible equilibria*, that is, those that may not prescribe equilibrium play after a subsequent state, we shall introduce the notion of *Markov Perfect Equilibria* (MPE).[4]

Definition2 *We say that a pair of strategies* $\left(\bar{\gamma}_1^T, \bar{\gamma}_2^T\right)$ *is called a* ***Markov Perfect Nash Equilibrium*** *(MPE) of the dynamic game if* ***for every feasible state*** $x_k \in S_k$ ***at time period*** $0 \leq k < T$, *we have for every feasible pair* $\left(\gamma_1^T, \gamma_2^T\right)$*:*

$$\sum_{t=k}^{T}\delta^t\, U_t^1\left(\bar{\gamma}_t^1(x_t), \bar{\gamma}_t^2(x_t)\right) \geq \sum_{t=k}^{T}\delta^t \cdot U_t^1\left(\gamma_t^1(x_t), \bar{\gamma}_t^2(x_t)\right)$$
$$\sum_{t=k}^{T}\delta^t\, U_t^2\left(\bar{\gamma}_t^1(x_t), \bar{\gamma}_t^2(x_t)\right) \geq \sum_{t=k}^{T}\delta^t \cdot U_t^2\left(\bar{\gamma}_t^1(x_t), \gamma_t^2(x_t)\right)$$

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[3] Also as for this, see Fudenberg and Tirole (1991), mentioned above.

[4] Technically, the reason why we consider "Markov", not "history dependent", strategy is to make heavy use of the monotonicity property. General (history dependent) strategies will generate the complexities of equilibrium and make the analysis messy. They do not seem to harmonize well with *Monotone Comparative Statics* in our context.

In words, $\bar{\gamma}^T$ is said to be a Markov Perfect Equilibrium (MPE) if and only if for every player $i = 1,2$ who would like to deviate from $\bar{\gamma}^T$ by playing another feasible strategy γ_1^T at every $x_k \in S_k$ at time period $0 \le k < T$, he or she would find no incentive to do so, as far as the other player follows $\bar{\gamma}^T$. For an infinite planning horizon, the above-presented definition applies by simply defining the payoff function to be the liminf of the payoffs for the finite horizon truncation of the game, formally.

2.3 Framework for Parametric Analysis.

Given evolving seasonality patterns, technological innovation, environmental changes etc, the actual shape of the renewal dynamics <u>may vary in time.</u> We assume that the universe of possible renewal dynamics is stationary and indexed by a parameter $p \in F$, where F is a finite, partially ordered set. Moreover, the indexing is such that it satisfies the following pointwise monotonicity property

$$p \ge q \Leftrightarrow f^p(x) \ge f^q(x)$$

In words, *the higher the parameter, the higher the renewal.* This implies that the following *Single Crossing Property* holds

$$\frac{\partial f^p(x)}{\partial x} \ge \frac{\partial f^q(x)}{\partial x}, \forall x$$

We make the convention that the *zero* parameter is associated with the *zero* renewal function.

Figure around here

A Preliminary Result by using 'Single Crossing Properties'[5] (Milgrom and Shannon (1994) and Edlin and Shannon (1998))

When $f^p(x)$ has *single crossing property* in p in that $\frac{\partial f^p(x)}{\partial x}$ is increasing in p, then the payoff function $U_i(e^1, e^2, x, p), i = 1,2$ has *Milgrom-Shannon Single Crossing Condition (MS-SCC)* in $-p$, that is, the marginal utility (marginal rate of substitution) $\frac{\partial U_i}{\partial e^i}, i = 1,2$ is monotonically non-increasing in the parameter p. And then, the best response $BR^i(e^j, x, p), i \ne j, i = 1,2$ is monotonically

[5] This is essentially the same as the *sorting condition* or "*Spence-Mirrlees* condition" suggested in application to the agent's utility function in the field of pioneering informational economics, except that recent 'Monotone Comparative Statics' does not *assume* the function to be *differentiable*. The key property is *Increasing (Decreasing) Difference*, which is weaker than the *Single Crossing Property*, requiring the <u>differentiability</u> of functions. See, Fudenberg and Tirole (1991) Chap.4, and Topkis (1998).

non-increasing in p for all e^j, x, and thus the equilibrium is also monotonically non-increasing in p for all x.[6]

Figure around here

Interpretation

When an agent marginally increases the extraction level e_i, the start of next period stock x_{t+1} will decrease greater in p than in q, with $p \geq q$, that is, $\left|-f_t^p\left(x_t - e_t^i - e_t^j\right)\right| > \left|-f_t^q\left(x_t - e_t^i - e_t^j\right)\right|$.

Thus, the marginal decrease in the next period utility will be greater in p than in q. Hence, the extraction level in this period e_t will be decreasing in p. In other words, when the state is q, which implies that the situation (e.g. land, common-pool resource, common fishery place) is harder to recover, the extraction level will be greater and the competition will be fiercer.[7]

2.4 Preliminaries before presenting our Theorems

A parameter profile for the $T-$planning horizon dynamic game is an element of the $T-$Cartesian product of F which we shall denote by F_T. Similarly, for the infinite horizon dynamic game, a parameter profile is simply an element of infinite Cartesian product of F, which we shall denote by F^∞. So, F_T and F^∞ are the set of $T-$horizon forecast, and the set of infinite horizon forecasts, respectively. Formally:

$$F_T = \prod_{t=0}^{T-1} F \qquad F^\infty = \prod_{t=0}^{\infty} F$$

We endow F with the discrete topology by means of the metric $\rho_t : F \times F \to \{0,1\}$ defined as follows: for $p_t, q_t \in F$, $\rho_t(p_t, q_t) = \begin{cases} 1 & if\ p_t = q_t \\ 0 & otherwise \end{cases}$. By means of this metric, we define a metric $d : F^\infty \times F^\infty \to R$ on F^∞ as follows. For $p = (p_0, p_1, \ldots..), q = (q_0, q_1, \ldots..) \in F^\infty$,

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[6] Edlin and Shannon (1998) extends the results of Milgrom and Shannon (1994) by imposing a stronger differential version of the single crossing property and arguing from first-order conditions. They replace *increasing (decreasing) differences* in Milgrom and Shannon (1994) by the marginal payoff $\partial U_i\left(e,(p,x)\right)/\partial e_i, i = 1,2$ *strictly increasing (decreasing) in the parameter* p. Then all selection of the optimal solutions are strictly increasing (decreasing), when they are in the interior. At this stage, we do not necessarily impose the differentiability of the payoff function $U(e,p)$.

[7] We could have an essentially same exercise, by using such a monotonicity that if the discount factor δ increases, then the exploitation (current consumption) level will be monotonically non- increasing in δ for every action by other player, and so the equilibrium will also be monotonically non-increasing.

$$d(p,q)=\sum_{t=0}^{\infty}\frac{\rho_t(p_t,q_t)}{2^t}$$

We remark that this metric induces the product topology, and that by Tychonoff's Theorem the space (F^∞,d) is compact.

For every $p \in F^\infty$, we shall denote by $\Pi_T^*(p)$ and $\Pi^*(p)$ the (possibly empty) set of MPE of the game with renewals indexed by p for the first T –periods and the (possibly empty) set of MPE for the Infinite Horizon game with renewal dynamics indexed by p.

As Fudenberg and Tirole (1991) explains, the condition of 'continuity at infinity' assures that events after t (for t large) have little effect, and one could expect that under the condition the sets of equilibria of finite-horizon and infinite-horizon versions of the 'same game' would be closely related, but it is not true that all infinite-horizon equilibria are limits of equilibria of the corresponding finite-horizon game. Hence, we need to make our standing assumptions[8], implied by the "Limit Theorem" by Fudenberg and Levine (1983), and also see Harris (1985).

Assumption 1: For every $p \in F_T$, any indexed collection $\{(\bar{\gamma}_1^{T,p},\bar{\gamma}_2^{T,p})\}_{T\in N}$ such that $(\bar{\gamma}_1^{T,p},\bar{\gamma}_2^{T,p})\in\Pi_T^*(p)$, $(\bar{\gamma}_1^{T,p},\bar{\gamma}_2^{T,p})\to(\bar{\gamma}_1^{p},\bar{\gamma}_2^{p})$ as $T\to\infty$ where convergence is with respect to the FLH topology[9], we have that $(\bar{\gamma}_1^{p},\bar{\gamma}_2^{p})\in\Pi^*(p)$.

We recall that two infinite horizon strategy pairs are close in the FLH topology if they prescribe the same contingent plan of actions for early stages of the game. For complete studies on sufficient conditions that imply Assumption 1, see Fudenberg and Levine (1983), Harris (1985), and Fudenberg and Tirole (1991). Formally,

Assumption 2: For every $p \in F^\infty, \Pi^*(p)$ is compact with respect to the FLH topology.

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[8] In single player version, we could assume that the limit of a converging sequence of finite horizon optimal solutions is an optimal solution for the infinite horizon problem.

[9] The notion of convergence related to the topology by FLH (Fudenberg-Levine (1983) and Harris (1985)) is the fact that $\gamma^{T,p}\to\gamma^{p}$ if and only if for all subgames, the sequence of histories induced by the T-horizon strategies $\gamma^{T,p}$ simultaneously converges in the discrete topology (they fully correspond) to the histories induced by the infinite horizon strategies γ^{p}. This convergence requires that for all t there exists a $T(t)$ such that for all $T \geq T(t)$ we have that the strategy combinations from the first period up to period t as prescribed by $\gamma^{T,p}$ and γ^{p} correspond.

3. Existence of "Solution Horizon" or "Early Turnpike"

Theorem 1 ***(Existence of Solution Horizon or 'Early Turnpike')***
Assuming that there exists a doubly indexed collection of Markov Perfect Equilibria:

$$\left\{\left(\bar{\gamma}_1^{T,p}, \bar{\gamma}_2^{T,p}\right)\right\}_{T\in N, p\in F_T}$$

such that the first time period outcomes are monotonically decreasing in $p \in F_T$*, i.e.*

$$p \geq q \Rightarrow \gamma_{0,p}^i(x_0) \leq \gamma_{0,q}^i(x_0) \qquad i = 1,2$$

then there exists an infinite horizon Markov Perfect Equilibrium $\left(\bar{\gamma}_1^p, \bar{\gamma}_2^p\right)$ *and a planning horizon* $\overline{T}$ *such that for* $T \geq \overline{T}$ *we have:*

$$\gamma_{0,p}^{i,T}(x_0) = \gamma_{0,p}^i(x_0)$$

Such horizon is called a ***"Solution" horizon*** *or in* ***"Early Turnpike"****.*

Proof. Let us now append <u>the *zero* strategy</u> to $\left(\bar{\gamma}_1^{T,p}, \bar{\gamma}_2^{T,p}\right)$ at period T. i.e.:

$$\left(\gamma_0^{i,T,p}, \gamma_1^{i,T,p} \ldots . \gamma_{T-1}^{i,T,p}, 0\right)$$

By the definition of MPE, we have that the extended pair must be an MPE to the $T+1-$planning horizon game with parameters $(p_0, p_1, \ldots, p_{T-1}, 0)$ and by the definition of the *zero* parameter:

$$(p_0, p_1, \ldots, p_{T-1}, 0) \leq (p_0, p_1, \ldots, p_{T-1}, p_T)$$

hence, by *the monotonicity*, it follows that:

$$\gamma_0^{i,T,p}(x_0) \geq \gamma_0^{i,T+1,p}(x_0)$$

By the compactness, the sequence $\left\{\gamma_0^{1,T,p}(x_0), \gamma_0^{2,T,p}(x_0)\right\}_{T\in N}$ must converge.

On the other hand, by assumption 1 (Fudenberg and Levine 1983, Harris 1985), the infinite extension:

$$\left(\gamma_0^{i,T,p}, \gamma_1^{i,T,p} \ldots . \gamma_{T-1}^{i,T,p}, 0, 0 \ldots\right) \in \Pi^*(p_1, p_2, \ldots, p_{T-1}, 0, 0, \ldots)$$

Moreover, by assumption 2 (compactness of the set of MPE for the infinite horizon games: Fudenberg and Levine 1983, Harris 1985), the extended collection has a converging subsequence. Let us denote by $\gamma^{i,p}$ the limit of such subsequence:

$$\lim_{k\to\infty} \gamma^{i,T_k,p} = \gamma^{i,p}$$

Now by convergence in the FLH topology:

$$\lim_{T \to \infty} \gamma_0^{i,T,p} = \gamma_0^{i,p}$$

Or equivalently: there exists a planning horizon $\overline{T}$ such that for $T \geq \overline{T}$ we have[10]:

$$\gamma_{0,p}^{i,T}(x_0) = \gamma_{0,p}^{i}(x_0) \qquad \blacksquare$$

In the proof of the above theorem, we have constructed a well-defined strategy $\gamma_0^{i,p}: F \to R^+$.
The next corollary establishes that **this strategy inherits <u>the monotonicity property</u>.**

Corollary 1 *Under the same assumptions of the above theorem, the strategies* $\gamma_0^{i,p}: F \to R^+ \; i = 1,2$ *are monotone decreasing and continuous.*

Proof. Let $p, q \in F$ and $p \geq q$. By the Solution Horizon Existence Theorem, there exist a planning horizon $\overline{T}^P$ such that for $T \geq \overline{T}^P$ for $i = 1,2$ we have:

$$\gamma_0^{i,T,p}(x_0) = \gamma_0^{i,p}(x_0)$$

Similarly, there exists a planning horizon $\overline{T}^q$ such that for $T \geq \overline{T}^q$, for $i = 1,2$, we have:

$$\gamma_0^{i,T,q}(x_0) = \gamma_0^{i,q}(x_0)$$

Let us set $\overline{T} \geq \max\{\overline{T}^p, \overline{T}^q\}$. By this choice, we have that for $T \geq \overline{T}$, by monotonicity property,

$$\gamma_0^{i,q}(x_0) \geq \gamma_0^{i,p}(x_0)$$

Moreover, since by using the triangle inequality

$$\left|\gamma_0^{i,p} - \gamma_0^{i,q}\right| \leq \left|\gamma_0^{i,p} - \gamma_0^{i,T,p}\right| + \left|\gamma_0^{i,T,p} - \gamma_0^{iT,q}\right| + \left|\gamma_0^{i,T,q} - \gamma_0^{i,q}\right|$$

for any $\xi > 0$, by continuity of $\gamma_0^{i,T,p}$ in $p \in F$ and the sequential construction of $\gamma_0^{i,p}$, there exists $\varepsilon > 0$ and T such that if $d(p,q) < \varepsilon$, then

$$\left|\gamma_0^{i,p} - \gamma_0^{i,q}\right| \leq \left|\gamma_0^{i,p} - \gamma_0^{i,T,p}\right| + \left|\gamma_0^{i,T,p} - \gamma_0^{i,T,q}\right| + \left|\gamma_0^{i,T,q} - \gamma_0^{i,q}\right| \leq \xi$$

Hence, the map $\gamma_0^{i,p}: F \to R^+$ is continuous. $\blacksquare$

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[10]Note that this is the result on the first (early) period equilibrium outcome, and that we imposed more restrictive conditions on the convergence, as mentioned in footnote 9. That's why we have the "<u>exact</u>" result. Otherwise, we could only state that *there exists an infinite horizon Markov Perfect Equilibrium* $(\overline{\gamma}_1^p, \overline{\gamma}_2^p)$ *such that* for every $\varepsilon > 0$, *there exists a planning horizon* $\overline{T}$, *such that for* $T \geq \overline{T}$ *we have (in an <u>approximate</u> form):*
$\left|\gamma_{0,p}^{i,T}(x_0) - \gamma_{0,p}^{i}(x_0)\right| < \varepsilon$

4. Existence of Forecast Horizon.

In this section we prove the existence of Forecast Horizons for the class of dynamic optimization problems considered by exploring the monotonicity properties of optimal solutions. For more detailed analysis on sufficient conditions for these monotonicity properties to hold, we refer the reader to Milgrom and Roberts (1994), Milgrom and Shannon (1994), Edlin and Shannon (1994)[11], Topkis (1998) and Athey (2001). Let us now state and prove the most important result.

Theorem 2 (***Existence of Forecast Horizon***) *Assuming that there exist a doubly indexed collection of Markov Perfect Equilibria:*

$$\left\{\left(\overline{\gamma}_1^{T,p},\overline{\gamma}_2^{T,p}\right)\right\}_{T\in N,\,p\in F_T}$$

such that the first time period outcomes are monotonically decreasing in $p\in F^T$ *i.e.:*

$$p\geq q\Rightarrow \gamma_{0,p}^{i}\left(x_0\right)\leq \gamma_{0,q}^{i}\left(x_0\right)\qquad i=1,2$$

then there exists an infinite horizon Markov perfect equilibrium $\left(\overline{\gamma}_1^{p},\overline{\gamma}_2^{p}\right)$ *and a planning horizon* $\overline{T}$ *such that for* $T\geq\overline{T}$ *and every* $q\in F^{\infty}$ *with* $p_t=q_t$ *for* $0\leq t\leq\overline{T}$, *we have:*

$$\gamma_{0,p}^{i,T}\left(x_0\right)=\gamma_{0,p}^{i}\left(x_0\right)=\gamma_{0,q}^{i}\left(x_0\right)$$

*Such horizon is called a **"Forecast "horizon.***

Proof. Let us **construct** a sequence of forecasts based on $p\in F$ as follows; we append a minimum forecast[12] $\underline{p}_T$ to the $T-$ truncation of $p\in F^{\infty}$, and obtain the forecast:

$$u\left(T\right)=\left(p_0,p_1\dots p_{T-1},\underline{p}_T,\underline{p}_{T+1}\cdots\right)$$

where clearly we have:

$$u(T)\rightarrow p\ \text{ as }\ T\rightarrow\infty \tag{1}$$

But by the Solution Horizon Existence theorem (Theorem1), we know that:

$$\lim_{T\to\infty}\left|\gamma_{0,p}^{i,T}-\gamma_{0,p}^{i}\right|=0 \tag{2}$$

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[11] We conjecture that the following theorem 2 could be understood as the one under a situation where the first period equilibrium outcome was not in the interior, but at a boundary, in connection with Edlin and Shannon (1998).

[12] By assumption, F is a finite set endowed with a particular ordering $\geq$, according to which there is a minimum $\underline{p}$ and a maximum forecast $\overline{p}$. We make the convention that $\underline{p}\geq 0$.

and by Corollary 1, the sequence $\left\{\gamma_0^{u(T)}\right\}$ is monotonically decreasing in T, i.e.:

$$\gamma_0^{u(T+1)} \le \gamma_0^{u(T)}$$

By compactness of the first period strategy space and continuity of the map $\gamma_{0,p}^i : F \mapsto R^+$:

$$\lim_{T\to\infty}\left|\gamma_{0,u(T)}^i - \gamma_{0,p}^i\right| = 0 \qquad (3)$$

So the results (2) and (3) ensure the existence of a large enough horizon $\overline{T}$ such that for $T \ge \overline{T}$:

$$\gamma_{0,p}^{i,T} = \gamma_{0,u(T)}^i = \gamma_{0,p}^i$$

Now let us consider $q \in F^\infty$ such that $p_t = q_t for 0 \le t \le \overline{T}$. Then, by *monotonicity*, it follows that for the choice of $\overline{T}$, for every $T \ge \overline{T}$:

$$\gamma_{0,u(T)}^i \ge \gamma_{0,q}^i \ge \gamma_{0,p}^i$$

Hence, $\gamma_{0,p}^i = \gamma_{0,q}^i$: in words, there exists an infinite horizon first period equilibrium outcome that is insensitive to parameter changes after time period $\overline{T}$ ■

5. Foundations, Applications, and A Robust Prediction for Existence of Forecast Horizon

5.1 Proof of the assumption part of Theorems 1 and 2

We proved theorems 1 and 2, under the statement that *Assuming that there exists a doubly indexed collection of Markov Perfect Equilibria:* $\left\{\left(\overline{\gamma}_1^{T,p}, \overline{\gamma}_2^{T,p}\right)\right\}_{T\in N, p\in F_T}$ *such that the first period outcomes are monotonically non-increasing in* $p \in F_T$*, i.e.* $p \ge q \Rightarrow \gamma_{0,p}^i(x_0) \le \gamma_{0,q}^i(x_0), i = 1,2$.

Here we shall give the classical, simple proof, borrowing from the **mechanism design** literature and under the assumption of *differentiability*.

Definition The function U_i satisfies the *sorting condition* if it is **twice differentiable** and either $\left(CS^+\right)\dfrac{\partial^2 U_i}{\partial p_t \partial e_t} \ge 0$ or $\left(CS^-\right)\dfrac{\partial^2 U_i}{\partial p_t \partial e_t} \le 0$. The sorting condition $\left(CS^+\right)$ means that a higher state variable (parameter) $p_t \uparrow$ makes a higher action $e_t \uparrow$ more desirable. And, conversely for $\left(CS^-\right)$.

Supplementary Theorem Suppose that our dynamic game satisfies the payoff separability and the *sorting conditions (single crossing properties)*. Then, the equilibrium Markov strategies

$\gamma_t(p_t), t = 0,1,2,...$ are *nondecreasing* under CS^+ or *nonincreasing* under CS^-.

Proof

The dynamic payoff function of each player $i = 1,2$ is written as

$U_i(e_t, p_t) + W_i(e_{t+1}, e_{t+2}, .., p_{t+1}, p_{t+2}, ...)$, where $e_t, e_{t+1}, ...$ are the vectors, consisting of the action pair in period $t = 0,1,2,...$

Fix Markov strategies for the other player and let

$$V_i(p_{t+1}, p_{t+2}, ...) \equiv W_i(p_{t+1}, e_{t+1}(p_{t+1}), p_{t+2}, e_{t+2}(f^{t+1}(e_{t+1}(p_{t+1})), p_{t+2}), ...)$$

denote **the continuation valuation (value function)** of player i for state variable (forecasts) $p_{t+1}, p_{t+2}, \cdots$..

Now consider the two possible states vector (forecasts), starting at date t, p_t and q_t, and let $e_{p_t} = \gamma^t(p_t)$ and $e_{q_t} = \gamma^t(q_t)$. By definition of equilibrium, player i prefers action e_{p_t} to action e_{q_t} when the state (forecasts) is p_t:

$$U_i(e_{p_t}, p_t) + V_i(f^{t+1}(e_{p_t}), p_{t+1}) \geq U_i(e_{q_t}, p_t) + V_i(f^{t+1}(e_{q_t}), p_{t+1})$$

Similarly, in state (forecasts) q_t, player i prefers action e_{q_t} to action e_{p_t}

$$U_i(e_{q_t}, q_t) + V_i(f^{t+1}(e_{q_t}), q_{t+1}) \geq U_i(e_{p_t}, q_t) + V_i(f^{t+1}(e_{p_t}), q_{t+1})$$

These inequalities are called the *incentive-compatibility constraints*. Adding them up, and separating the current payoffs and the continuation values, we have:

$$\begin{aligned}&\{U_i(e_{p_t}, p_t) + U_i(e_{q_t}, q_t) - U_i(e_{q_t}, p_t) - U_i(e_{p_t}, q_t)\}\\&+\{V_i(f^{t+1}(e_{p_t}), p_{t+1}) + V_i(f^{t+1}(e_{q_t}), q_{t+1}) - V_i(f^{t+1}(e_{q_t}), p_{t+1}) - V_i(f^{t+1}(e_{p_t}), q_{t+1})\} \geq 0\end{aligned}$$

This can be rewritten as

$$\int_{e_{q_t}}^{e_{p_t}} \int_{q_t}^{p_t} \frac{\partial^2 U_i}{\partial z \partial y} dz dy + \int_{f^{t+1}(e_{q_t})}^{f^{t+1}(e_{p_t})} \int_{q_{t+1}}^{p_{t+1}} \frac{\partial^2 V_i}{\partial z \partial y} dz dy \geq 0 .$$

Now, if $p_t \geq q_t$ in the partial order and CS^+ holds, then such a *monotone path* as $e_{p_t} \geq e_{q_t}$ and

$e_{p_{t+j}} \geq e_{q_{t+j}}, t = 0,1,2,...; j = 1,2,....$ satisfies this inequality. On the other hand, other (potentially many) paths implemented by Nash equilibra in Markov strategies can also satisfy this inequality , but are dominated by the monotone path, due to the requirement of "*perfect-ness*". Similarly, we have

$e_{p_t} \leq e_{q_t}, t = 0,1,2,....$ if CS^- holds. ∎

5.2 More Foundation on the Crucial Assumption: Monotonicity Properties of Optimal Solutions

We have made heavy use of monotonicity properties of optimal solutions to obtain early turnpike results and existence of Forecast Horizon. Though we assumed these properties in theorems 1 and 2, according to recent results by 'Monotone Comparative Statics', it is known that as a sufficient condition for these monotonicities to hold endogenously, payoff functions must satisfy the so-called 'Single crossing properties' developed by Milgrom and Shannon (1994), Edlin and Shannon (1998), and Topkis (1998).[13] We also gave a simple proof on how the monotonicity of equilibrium behavior could be derived, depending on the sorting/single crossing condition, in section 5.1. In our model of exploitation of common pool resources, such as a fisheries model where two agents choose in each period how much harvest to extract, the actions by both players e_t^1, e_t^2 are *strategic substitutes*, which means that the utility function $U_t(e_t^1, e_t^2)$ satisfies *decreasing differences* (ID) in e_t^1 and e_t^2, or the *cross-partial derivatives* of the players' per-period payoffs with respect to e_t^1 and e_t^2 are negative. Hence, according to a general principle by Topkis (1998), our paper so far focuses on a setting of *submodular* games.

[13]Athey (2001) proves a theorem for the existence of pure strategy Nash equilibrium (PSNE) in games of incomplete information, such as auctions and noisy signaling games. She shows that if a player's optimal action is non-decreasing in its "type" (marginal cost, valuation), then a PSNE exists under very general conditions, and has established a close and previously unrecognized link between Monotone Comparative Statics (MCS) and the existence of PSNE.

5.2.1 Environmental Substitutability/Complementarity

Can we assert that in general framework of submodular games or in supermodular games, the same results of obtaining both early turnpike and existence of Forecast Horizon will hold? That is, though it is not doubtful that the most important and crucial property is **the *monotonicity* of optimal solutions or equilibrium behaviors**, don't we depend the results on whether the games are supermodular or submodular, or in other words, in addition to the single crossing property , is it important whether the sign of the sorting condition is $\frac{\partial^2 U_t}{\partial p_t \partial e_t} \geq 0$ $(CS^+ : \text{Environmental Complementarity})$ or $\frac{\partial^2 U_t}{\partial p_t \partial e_t} \leq 0$ $(CS^- : \text{Environmental Substitutability})$? (CS^+) means that a higher state variable (parameter p_t) makes a higher action (e_t or equivalently γ_t) more desirable, that is, when the state variable increases, the marginal returns to a player from increasing his strategy increases too, as is not the case in the main text. Our prediction is that our result does not depend on 'Environmental Substitutability/ Complementarity', but more generally holds, that is, *robust.* From the proofs in the original model, we can deduce that we could obtain both theorems 1and 2. The following is the sketch.

Now, suppose that (CS^+) holds. Then, from the proof of theorem 1, it follows that

$\gamma_0^{i,T,p}(x_0) \leq \gamma_0^{i,T+1,p}(x_0)$ from (CS^+) and the *monotonicity*. Then, due to the assumption 2, which is a result by Fudenberg and Levine (1983) and Harris (1985), the sequence $\{\gamma_0^{1,T,p}(x_0), \gamma_0^{2,T,p}(x_0)\}_{T \in N}$ would have an *accumulation point*, which is equivalent to the *convergen*ce point of the sequence and <u>due to the assumption 1</u> a first period Markov Perfect Equilibrium outcome of the infinite horizon game. Similarly, in theorem 2, supposing that (CS^+) holds, by the Solution Horizon Existence theorem (Theorem1), we know that: $\lim_{T\to\infty} \left|\gamma_{0,p}^{i,T} - \gamma_{0,p}^{i}\right| = 0$. And since $u(T) = (p_0, p_1 \dots p_{T-1}, \underline{p}_T, \underline{p}_{T+1} \dots)$ is a minimum forecasts appended to the $T-$ truncation of $p \in F^\infty$, we have $\gamma_0^{u(T)} \leq \gamma_0^{u(T+1)}$, and by compactness of the first period strategy space and continuity of the map $\gamma_{0,p}^{i} : F \mapsto R^+$, we have $\lim_{T\to\infty} \left|\gamma_{0,u(T)}^{i} - \gamma_{0,p}^{i}\right| = 0$. Hence, for $T \geq \overline{T}$ (a large enough horizon), $\gamma_{0,p}^{i,T} = \gamma_{0,u(T)}^{i} = \gamma_{0,p}^{i}$.

Further, from (CS^{+}) and the *monotonicity,* $\gamma^{i}_{0,p} \geq \gamma^{i}_{0,q}$ when $p \geq q$. Then, for $p_t = q_t, 0 \leq t \leq \overline{T}$, we have $\gamma^{i}_{0,p} \geq \gamma^{i}_{0,q} \geq \gamma^{i}_{0,u(T)}$, and so we find that $\gamma^{i}_{0,q} = \gamma^{i}_{0,p}$ does hold, for $T \geq \overline{T}$. From these observations, whether the sign on the effect of environmental parameter p_t is $(CS^{+})\dfrac{\partial^2 U_t}{\partial p_t \partial e_t} \geq 0$ or $(CS^{-})\dfrac{\partial^2 U_t}{\partial p_t \partial e_t} \leq 0$ **DOES NOT MATTER**. When $(CS^{+})\dfrac{\partial^2 U_t}{\partial p_t \partial e_t} \geq 0$ holds, there still exists an infinite horizon first period equilibrium outcome insensitive to parameter changes after a time period $\overline{T}$, i.e. 'forecast horizon' exists. And, this existence result most critically depends on **the *monotonicity* of optimal solutions / equilibrium behaviors,** not on the sign of the sorting condition ('Environmental Substitutability/ Complementarity').

5.2.2 Strategic Substitutability/Complementarity

The above observations are made about ***environmental*** **Substitutability/Complementarity**. Next, is it important for existence of forecast horizon whether the games are decreasing difference or increasing difference in ***strategic variables,*** *that is,* ***strategic*** **substitutes or complements** $\Leftrightarrow \dfrac{\partial^2 U}{\partial e_1 \partial e_2} \leq 0$ or $\dfrac{\partial^2 U}{\partial e_1 \partial e_2} \geq 0$? We shall start from investigating an example of games with strategic complementarity.

5.2.2.1 Diamond-type Search with learning

Consider the following stage game originally due to Diamond (1982) and presented by Milgrom and Roberts (1990) to illustrate the theory of supermodular games: there are n players who exert effort searching for trading partners. $e_i \in [0,1]$ denotes the effort of player i. A coefficient α is a measure of the productivity of search. A player exerting an effort e incurs a cost $C(e)$, where $C(.)$ is a smooth, increasing function. Now, the payoff to player i is defined by:

$$\pi_i(e) = \alpha \cdot e_i \sum_{j \neq i} e_j - C(e_i)$$

First, check **the *complementarity* assumption**. As one would expect from a model where agents search for trading partners among each others, the payoff π_i exhibits *strategic complementarities* (SC) between agent i's effort to find a partner and the other agents' efforts. It is easily verified from the fact that

$\frac{\partial \pi_i}{\partial e_i} = \alpha \sum_{j \neq i} e_j - \frac{\partial C(e_i)}{\partial e_i}$ and $\frac{\partial^2 \pi_i}{\partial e_i \partial e_j} = \alpha > 0$. Also, the higher the coefficient α_t, the higher the marginal return of any player's effort of search, from $\frac{\partial^2 \pi_i}{\partial e_i \partial \alpha} = \sum_{j \neq i} e_j \geq 0 \quad \forall i = 1...n, i \neq j$. This agrees with the interpretation that α is an index of the productivity of the search. Thus the payoff π_i has ***increasing differences*** in e_i and (α, e_{-i}), and also is supermodular in e_i, as e_i is one-dimensional.

Now, a stochastic game is constructed with this search model. Assume that traders meet for a **finite or infinite** number of periods, and that the game is **parameterized** in the following way: one state variable is α_t, and the actions are $e_{i,t}$ for all $i = 1, ..., n$. Payoff to trader i is then:

$$\pi_i(e_t, \alpha_t) = \alpha_t \cdot e_{i,t} \sum_{j \neq i} e_{j,t} - C_t(e_{i,t})$$

Given the current level α_t of the search productivity and the search effort $e_{i,t}, i = 1, ..., n$ at time period t, the dynamics (the evolution of the state variable) is modeled by[14]

$$\alpha_{t+1} = f_t(\alpha_t + e_{1,t} + ... + e_{n,t}) \qquad t = 0,1,2,...$$

Given evolving seasonality patterns, technological innovation (improvement), changes in regulation etc, the actual shape of the dynamics may vary in time. We assume that the universe of possible dynamics is

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[14]**Remark: non-stationary distribution function**

Assume that α_t is subject to *a stochastic evolution* in the interval $[1,2]$ through learning from the players: the higher the effort made by the traders to find partners in period t, the more they know about the environment and the more productive their search efforts are therefore in expected terms, and thus the higher the coefficient α_{t+1} in period $t+1$. Formally, let the distribution function for α_{t+1} given (α_t, e_t) be:

$$\forall u \in [1,2], \qquad G_t(u; \alpha_t, e_t) = (u-1)\left[1 - \frac{\alpha_t + e_{1,t} + ... + e_{n,t}}{2+n}(u-2)^{p_t}\right]$$

$G_t(u; \alpha_t, e_t)$ is plotted for various numerical values of (n, α_t, e_t). This process does not necessarily generate *increasing sequences of productivity indexes* $\{\alpha_t\}_{t=0,1,2,....}$ It may be the case that the players forget part of what they knew about the environment. In this setting, the larger the parameter p_t, the higher the next period coefficient α_{t+1} in expected terms. Hence, the essence is the same.

stationary and indexed by a parameter $p \in F$, where F is a finite, partially ordered set. Moreover, the indexing is such that it satisfies the following pointwise monotonicity property

$$p \geq q \Leftrightarrow f^p(\alpha) \geq f^q(\alpha)$$

In words, *the higher the parameter, the higher the productivity of search.* This implies that the following *Single Crossing Property* holds $\quad \dfrac{\partial f^p(\alpha)}{\partial \alpha} \geq \dfrac{\partial f^q(\alpha)}{\partial \alpha}, \forall \alpha$

Now, the payoffs for the Infinite Horizon Game are:

$$\Pi_i = \sum_{t=0}^{\infty} \delta^t \cdot \pi_t^i\left(e_{1,t},\dots,e_{n,t}; p_t, \alpha_t\right) \qquad i = 1,..,n$$

When $f^p(\alpha)$ has *single crossing property* in p in that $\dfrac{\partial f^p(\alpha)}{\partial \alpha}$ is non-decreasing in p, then the payoff function $\Pi_i\left(e_1,..,e_i,..,e_n;\alpha,p\right), i = 1,\dots,n$ has *Milgrom-Shannon Single Crossing Condition (MS-SCC)* in p, that is, the marginal profit (marginal rate of substitution) $\dfrac{\partial \Pi_i}{\partial e_i}, i = 1,\dots,n$ is monotonically non-decreasing in p. Then, **the best response** $BR^i\left(e_{-i};\alpha,p\right), i = 1,\dots,n$ is also non-decreasing in p for all e_{-i}, α, and **thus the equilibrium** is also monotonically non-decreasing in p for all α.

An interpretation

When an agent i marginally increases the effort level e_i, the start of next period stock α_{t+1} will increase greater in p than in q, with $p \geq q$, that is, $\partial f^p\left(x_t + e_{i,t} + e_{-i,t}\right)/\partial e_{i,t} >$ $\partial f^q\left(x_t + e_{i,t} + e_{-i,t}\right)/\partial e_{i,t}$. Thus, the marginal increase in the next period profit will be greater in p than in q, from the form of the payoff function $\pi_i\left(e_t,\alpha_t,p_t\right) = \alpha_t \cdot e_{i,t} \sum_{j \neq i} e_{j,t} - C_t\left(e_{i,t}\right)$.

Hence, the search effort level in this period e_t will be also non-decreasing in p.

After this, the same argument as in the Common Pool Resource game would follow. The monotonicity of optimal solutions: $p \geq q \Rightarrow \; e^i_{0,p}\left(\alpha_0\right) \geq e^i_{0,q}\left(\alpha_0\right), i = 1,\dots,n$ would result in the two theorems, from the argument in section 5.2.1. Hence, we have the following prediction.

A Robust Prediction

For games with both **Strategic Substitutes and Complements**, there exist both an early turnpike and a forecast horizon, if there exists a state parameter, which satisfies a **'sorting condition' (environmental substitutability (complementarity))**, in that the payoff functions π_i and Π_i have *decreasing (increasing) differences* in own action and the state parameter, or under the differentiability, the marginal payoff is monotonically non-increasing(decreasing) in the state parameter. That is, our results on the existences of an early turnpike and a forecast horizon are 'robust'.

5.2.2.2 On Multiple Equilibria

In games with *strategic complementarities*, it is known that there may often exist multiple equilibria. In the analysis of Diamond (1982), Milgrom and Roberts (1990) considers a parameterized class of super-modular games, where the payoff functions $\pi_i(e_i, e_{-i}; p, \alpha)$ have *Increasing Difference in* (e_i, α). Then, both the highest and the smallest Nash equilibria $\overline{e}(\alpha)$ and $\underline{e}(\alpha)$ are non-decreasing in α. Nevertheless, we will still obtain **the 'existence' result of 'Forecast Horizon'**, **on each of both the highest and the smallest equilibria**, when the payoff function $\Pi_i(e_i, e_{-i}; p, \alpha)$ has *Increasing Difference in* (e_i, p), in summery, satisfies "Single Crossing Properties" and "Monotonicities", and so $p \geq q \Rightarrow e^i_{0,p}(\alpha) \geq e^i_{0,q}(\alpha)$, $i = 1, \ldots, n$.

Figure around here

5.3 An Economic Application of Results: Tournaments and Hold-up

We shall pick up two more interesting examples that we can apply the results of existence of early turnpike (solution horizon) and Forecast Horizon.

First, we consider an implication to **rank-order tournaments** literature, which include Lazear and Rosen (1981) and Rosen (1986) among others. Our model can be interpreted in such a way that two fishermen have the relative status consideration $z_i \equiv z(e_i - e_j)$, and z_i depends on performance differences between the two fishermen, i.e. $e_i - e_j$. We assume that social esteem increases as performance differences enlarge (i.e., z_i is an increasing function of performance difference). That is, a positive (negative) difference in performance signifies a higher (lower) status, which generates positive (negative) feelings arising from self-esteem (feeling ashamed). Social esteem (relative status, rank consideration) mechanism could be interpreted as "tournaments". In tournaments with no uncertainty, the payoff function of each player is known to be "discontinuous at ties", which brings about the non-existence problem of equilibrium. In order to rule out the existence problems associated with "discontinuity at ties", as is usual in tournament models, we approximate such discontinuous function with a "smooth" continuous function, by introducing some noise factors. Hence, $z_i \equiv z\left(e_i - e_j\right)$ is a reduced form function, given $e_i - e_j$, after such procedure.

Now, we assume that $z'\left(e_i - e_j\right) > 0, \forall e_i - e_j \in \Re$, thus $z'\left(0\right) > 0$, and that $z\left(0\right)$=constant.

The (static) payoff function for i is

$$U_i\left(e_i, z\left(e_i - e_j\right)\right) \equiv U_i\left(e_i, e_j; z\right), \text{ with } \frac{\partial U_i}{\partial e_i} > 0, \frac{\partial^2 U_i}{\partial e_i^2} < 0, \frac{\partial U_i}{\partial e_j} < 0, \frac{\partial^2 U_i}{\partial e_j^2} < 0, \text{ and}$$

$$\frac{\partial^2 U_i}{\partial e_i \partial e_j} = \frac{\partial^2 U_i}{\partial e_j \partial e_i} < 0, \text{ and} \frac{\partial^2 U_i}{\partial e_i \partial z} > 0, \frac{\partial^2 U_i}{\partial e_j \partial z} < 0.$$

Next, a dynamic game is constructed. Assume that the fishermen compete for a **finite or infinite** number of periods. Let $E_{i,t-1} \equiv \sum_{k=0}^{t-1} e_{i,k}, i = 1,2$ the accumulated performance at the beginning of time t of the fisherman i, and $dK_{i,t} = E_{i,t-1} - E_{j,t-1}, i, j = 1,2, i \neq j$, is the accumulated difference at the beginning of time period t. There is no depreciation.

Given the current difference level $dK_{i,t} = E_{i,t-1} - E_{j,t-1}$ and the effort $e_{i,t}$ at time period t,

We can define the function $z_i^p = z^p\left(dK_{i,t} + \left(e_{i,t} - e_{j,t}\right)\right)$, $i, j = 1,2, i \neq j$ which implies for instance the winning probability given $dK_{i,t} + \left(e_{i,t} - e_{j,t}\right)$ times the prize for the time period t tournaments. As for this z^p function, we keep ruling out the problem of "discontinuity at ties". That is, this is a "smooth" continuous function.

Let us examine Markov strategies as before. Markov perfect strategy is defined by considering subgames, given every feasible state (current state) $dK_{i,t}$ (note that $dK_{1,t} = -dK_{2,t}$) at time period $0 \leq t < T$. (Histories do not matter.)

The important point is that the actual shape of the z^p function may vary in time. The universe of possible dynamics is stationary and indexed by a parameter $p \in F$, where F is a finite, partially ordered set. The indexing satisfies the point-wise monotonicity:

$$p \geq q \Leftrightarrow z^p\left(dK_{i,t} + \left(e_{i,t} - e_{j,t}\right)\right) \geq z^q\left(dK_{i,t} + \left(e_{i,t} - e_{j,t}\right)\right),$$

$$\forall dK_{i,t} + \left(e_{i,t} - e_{j,t}\right), i \neq j, i = 1,2, \forall 0 \leq t < T$$

In words, *the higher the parameter, the higher the function* z. This implies that the following *Single Crossing Property* holds $\dfrac{\partial z^p(x)}{\partial x} \geq \dfrac{\partial z^q(x)}{\partial x}, \forall x$

Then, as Milgrom and Shannon (1994) and Edlin and Shannon (1998) shows, the payoff function $U_i\left(e_i, e_j; z\right)$ has *Single Crossing Condition (SCC)* in p. That is, the marginal profit (marginal rate of substitution) $\dfrac{\partial U_i}{\partial e_i}, i = 1,2$ is *monotonically non-decreasing* in p. And then, the Best Response $BR^i\left(e_j, p\right), i \neq j, i = 1,2$ is *non-decreasing* in p for all e^j and thus **the equilibrium is also monotonically non-decreasing in** p. This is a game with Strategic Substitutes (SS) and Environmental Complements (EC).

Then, in the same way, we can show that there exists a doubly indexed collection of Markov Perfect Equilibria $\left\{\left(\bar{\gamma}_1^{T,p}, \bar{\gamma}_2^{T,p}\right)\right\}_{T \in N, p \in F_T}$ such that the first time period outcomes are monotonically increasing in $p \in F_T$, i.e., $p \geq q \Rightarrow e^i_{0,p} \geq e^i_{0,q}, i = 1,2$, by making heavy use of the *monotonicity property*. Due to our theorems, we would have the following insights. Finite horizon tournaments can induce the same first period equilibrium outcomes as infinite horizon tournaments (Theorem1), and the insensitivity of first period equilibrium actions with respect to *parameter changes after forecast horizon* (Theorem 2) seems well fitted to our common sense that the players compete with each other in a forward-looking manner, but do not take "hyper-rational" behavior, as required theoretically in an infinite horizon game.

The second example is **a model of Hold-up**. There are two parties: Buyer B and Seller S. The two parties meet, ex post leading to a bilateral monopoly. B invests e_B, and S invests e_S. Ex post renegotiation surplus is $R(e_B) - C(e_S)$, where $R'(e_B) > 0,\ C'(e_S) < 0$. Ex post they renegotiate efficiently under symmetric information, dividing the renegotiation surplus 50/50 (Nash Bargaining Solution). Then, they choose ex ante investments

$$e_B^0 \in \arg\max_{e_B} \frac{1}{2}[R(e_B) - C(e_S)] - e_B$$

$$e_S^0 \in \arg\max_{e_S} \frac{1}{2}[R(e_B) - C(e_S)] - e_S$$

We will have underinvestment result, since each party internalizes only 50 % of its contribution to total surplus, while bearing all investments. Now, we consider state x_t as the common capital stock (specific skill) at time t, to which both parties can access, and the state in the next period x_{t+1} is given by a time dependent dynamic capital accumulation function $x_{t+1} = f^t\left(x_t + e_t^B + e_t^S\right)$. In this set up, formally, we define a function $\phi : E \times S \to \mathbb{R}$, where $E, S \subset \mathbb{R}$ has *Increasing Differences (ID)* if $\phi(e'', x) - \phi(e', x)$ is nondecreasing in $x \in S$, for all e'', e', such that $e'' > e'$. The intuition is that higher x (capital stock, skill) induces the benefits of raising e_B, e_S. This property is also called *supermodularity*. A key result in monotone comparative statics is that when the objective function satisfies *Increasing Differences (ID)*, maximizers are nondecreasing in the parameter value. So, due to the theorem of Topkis, supposing that ϕ has *ID*, $x'' > x'$ and $E(x) = \arg\max_{e \in E} \phi(e, x)$, then for any $e' \in E(x')$ and $e'' \in E(x'')$, $e'' > e'$ or $e' \in E(x'')$ and $e'' \in E(x')$. Seeing that by assumption $\phi_B(e_B, x) = \frac{1}{2}R(e_B, x) - e_B$ and $\phi_S(e_S, x) = -\frac{1}{2}C(e_S, x) - e_S$ have *Increasing Differences (ID)*, Topkis's Theorem implies monotonicity properties of optimal solutions: $e_B' \in E(x') \le e_B'' \in E(x'')$, where $x'' > x'$. Now, we shall introduce a dynamics, in which the actual size of renegotiation gain may vary in time, that is, the renegotiation gain is $(1-p)\left[R(e_B) - C(e_S)\right]$, where $p \in F$ is an indexed parameter and F is the universe of possible dynamics.[15] The indexing satisfies the *single crossing property*, in the sense that

$$p \ge q \Leftrightarrow \frac{\partial \phi_B^p}{\partial e_B} = \frac{1-p}{2}R'(e_B, x) - 1 \le \frac{1-q}{2}R'(e_B, x) - 1 = \frac{\partial \phi_B^q}{\partial e_B} \qquad \forall e_B, x$$

$$p \ge q \Leftrightarrow \frac{\partial \phi_S^p}{\partial e_S} = -\frac{1-p}{2}C'(e_S, x) - 1 \le -\frac{1-q}{2}C'(e_S, x) - 1 = \frac{\partial \phi_S^q}{\partial e_S} \qquad \forall e_S, x$$

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[15] In Markov Perfect Equilibria, strategies are restricted to be a function only of the current levels of "payoff relevant" state variables. In the two examples, $p, q \in F$ are payoff relevant state variables, which determine the correspondence to either current demand or cost conditions.

Then, using our results, we can obtain the monotonicity of the first period equilibrium outcomes

$$p \geq q \Rightarrow e^i_{0,p}(x_0) \leq e^i_{0,q}(x_0), i = B, S$$

Thus, we can apply the main results of our paper to this setting of Hold-up. That is, repeating the setting of Hold-up infinitely, we obtain the existence of "early turnpike" and "forecast horizon", and in addition we could compute explicitly the first period equilibrium outcome of a nonstationary dynamic version of Hold-up, by solving a finite horizon version of the game by truncating at the early turnpike (forecast horizon). This might make possible a new result in the context of dynamic Hold-up, such as a characterization of capital accumulation level and its comparison under different institutions.[16][17]

6. Conclusion.

We have applied the monotonicity properties of optimal solutions by *Monotone Comparative Statics* literature to a context of nonstationary dynamic games to prove "early turnpike results" and the existence of a "forecast horizon". Interestingly enough, the existence of such horizon indicates that after all, "Backward induction" can be performed for infinite horizon games, in the sense that, to solve for the first period equilibrium, one may consider the finite horizon dynamic game whose planning horizon is exactly the forecast horizon. Additionally, the result of the existence of a "forecast horizon" implies that for a sufficiently long horizon, we obtain *strategic decoupling,* i.e. first period equilibrium outcomes are insensitive to parameter changes *after* that horizon. Hence, theoretically, this result can be used to argue that the concept of *subgame perfect* equilibrium does not require "hyper-rational" behavior by the players engaged in an infinite horizon game, since the first stage game gets *decoupled* from the tail effects.

Jehiel (1995) views as bounded rationality as a limited ability to forecast the future , and shows that in two-player infinite-horizon alternating-move games, a limited forecast (n_1, n_2) -equilibrium exists, where (1) player i chooses actions, according to his n_i -length forecasts so as to maximize the average payoff over the forthcoming n_i periods, and (2) players' equilibrium forecasts are correct. In his paper, limited horizon forecast is a choice variable by players and, as he points out, his solution has no relationship with Markov Perfect Equilibrium (MPE) which we adopt as equilibrium concept.

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[16] As for a literature that considers both Tournaments and Hold-up, see Konishi, Okuno-Fujiwara, and Suzuki (1996). We used to try to characterize equilibrium capital accumulation, in a dynamic context of Hold-up, by seeking for the early turnpike and solving a finite horizon version of the game by truncating at the forecast horizon (early turnpike).

[17] Other applications will include an alternative offer bargaining game ala Rubinstein. Consider the parameters on the probability of break down when the receiver rejects the offer, $p = (p_0, p_1, ...)$, $q = (q_0, q_1, ...)$. Then, we will have the same result, such that $\gamma^T_{0,p} = \gamma_{0,p} = \gamma_{0,q}$ for $T \geq \overline{T}$, where $\gamma_{0,p}$ and $\gamma^T_{0,p}$ are the offers at the first period in the infinite and finite horizon game, respectively, with parameter p, q .

While, in our paper, the focus is a relationship between monotonicity of optimal solutions and existence of forecast horizon. Our result shows that under certain conditions, for a sufficiently long horizon, first period equilibrium outcomes are insensitive to parameter changes *after* that horizon, in other words, there is a *bounded planning horizon* in the sense that the sequence of parameters after that horizon does not affect the equilibrium play in the first period. We could interpret this result such that players only foresee the future *up to forecast horizon*. It could be said to be a different derivation from Jehiel (1995) of human being's *bounded rationality*.[18] The result also suggests a computational procedure to solve infinite horizon nonstationary games by means of a "rolling horizon " procedure, whereby an each iteration we solve a finite horizon version of the game by truncating at the forecast horizon.

[18] This may be connected to an *endogenous* derivation of *incomplete contracts*, in that first period equilibrium outcomes (contract agreements) are *insensitive to far future parameter changes (events), in other words, non-state-contingent.*

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