Undertaking nonharmful or harmful public projects through unit-by-unit contribution: Coordination and Pareto efficiency

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Abstract

We examine in detail the implementation of a project that is nonharmful for all agents as well as a project that is harmful for some agents through a *unit-by-unit contribution mechanism*. For a project that is nonharmful for all agents, efficient implementation is supported at one regular Nash equilibrium and several refined Nash equilibria that are stable against coalition deviations. In this sense, this mechanism works well. On the other hand, when the project is harmful for some agents, this mechanism may not have a Nash equilibrium with efficient implementation of the project. Even when such a Nash equilibrium exists, it may not be selected by any of the refined Nash equilibria. Thus, in this case, this mechanism does not work. Our result shows that the merit of the unit-by-unit contribution mechanism reported in the literature is partially extensible to the implementation of a public project.

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1 Introduction

We consider a public project implementation through a *unit-by-unit contribution mechanism*. We investigate in detail the implementation of a project that is nonharmful for all agents as well as a project that is harmful for some agents. We examine under what conditions the project is undertaken Pareto-efficiently through the unit-by-unit contribution mechanism.

The unit-by-unit contribution mechanism is introduced to provide a discrete pure public good in integer units. As in a standard case of public-good provision in nonnegative real numbers, voluntary public-good provision in nonnegative integer units suffers from the free-rider problem, so that the public good is not supplied Pareto efficiently.[1] One of the solutions to this problem is to construct public-good mechanisms. To solve the free-rider problem of an integer-unit public good, Bagnoli and Lipman (1989) introduce a unit-by-unit contribution mechanism. Later, Branzei et al. (2005) introduced another mechanism, which is a bit different from, but essentially the same as, Bagnoli and Lipman's (1989) mechanism[2] and applied it to a public-good problem that is different from the Bagnoli and Lipman (1989) problem. Their mechanisms to solve the problem are based on the idea that the level of public-good provision is decided through a "unit-by-unit" process. In their mechanisms, agents are asked to make marginal contributions to every one-unit increase in the public good. Based on the contributions, starting from the first unit of the good, the quantity increases by one unit as long as the sum of the marginal contributions to a one-unit increase covers its marginal cost. Bagnoli and Lipman (1989) and Branzei et al. (2005) show that the unit-by-unit contribution mechanism has a Nash equilibrium at which the public good is provided Pareto-efficiently. Moreover, they show that although this mechanism may have other Nash equilibria at which the public good is provided Pareto-inefficiently, some refinements of Nash equilibria single out the Nash equilibria with efficient provision of the public good. In this sense, the mechanism solves the free-rider problem of the provision of an integer-unit public good.[3]

We could say that this mechanism is based on a "simple" rule: whether the public good increases by one unit depends only on the relationship between the marginal contributions to and the marginal cost of this increase and the payment from each agent is the sum of her announced marginal contributions to each unit. Moreover, we could say that this mechanism is "suitable" in the provision of an integer-unit public good because it utilizes a discrete structure of an integer-unit public good. Because of this simplicity and suitability, it seems to have some applicability to the implementation of public projects in the real world. Hence, it would be important to know how this mechanism works in the provision of various public projects.

[1] For the voluntary provision of an integer-unit public good, see, for example, Bagnoli and Lipman (1989, p. 591, last paragraph), Gradstein and Nitzan (1990), and Shinohara (2009).

[2] See a detailed explanation of this point in Section 2.

[3] To be precise, Bagnoli and Lipman (1989) use a refinement of trembling-perfect Nash equilibria and Branzei et al. (2005) use a strong Nash equilibrium (Aumann, 1959). Their refinement concepts are completely different. They prove that payoffs attained at those refined Nash equilibria coincide with the core of a cooperative game. We also use several refinements of Nash equilibria based on coalition formation, including the strong Nash equilibrium.

However, this mechanism has been tested under limited situations in the literature. Bagnoli and Lipman (1989) and Branzei et al. (2005) assume that agents have a quasi-linear utility function with respect to a private good and benefits from a public good are measured in terms of the private good. Bagnoli and Lipman (1989) assume that agents' benefit functions from the public good are increasing and strictly concave in its level, which are seemingly standard conditions for public good provision. On the other hand, Branzei et al. (2005) assume that each agent has a threshold level of the public good and he receives a positive constant benefit if and only if the public good is provided at the threshold level or higher. How this mechanism works has not been clarified in the implementation of public projects that cannot be captured by those benefit structures.

Moreover, when it comes to public projects in the real world, they are sometimes harmful in the sense that raising the level of a public project may decrease someone's benefits. For example, consider the construction of a high-speed railway (HSR) network such as the *Shinkansen* bullet-train projects in Japan. This project connects Tokyo (the capital city) to the peripheral cities with HSR networks, which have been extended sequentially.[4] It is said that this extension has two sides: it may stimulate the local economies since tourism is promoted and some companies in the capital city establish branch offices in the local cities. On the other hand, it may create disadvantages such as outflow of population from local cities. In reality, these positive and negative sides would determine benefits of peripheral cities. Some empirical studies show that extension of the HSR network does not necessarily benefit peripheral cities.[5] If we interpret this extension as an increase in the project level, some agents might lose their benefits from the project by the increase. This shows public project effects that cannot be captured by the benefit structures of earlier studies. When examining the applicability of the unit-by-unit contribution mechanism to the implementation of real-world public projects, we need to consider the case in which a public project is harmful for some agents. However, this has not been considered in the literature.

In order to examine applicability of the unit-by-unit contribution mechanism, we need to introduce a framework that can capture as many public projects as possible. We introduce two types of public projects—one is "nonharmful" for all agents and the other is sometimes "harmful" for some agents—and examine the implementation of each public project through the unit-by-unit contribution mechanism. Our aim is to clarify to what extent this mechanism achieves efficient public project implementation in each case .

First, a project is defined to be *nonharmful for all agents* if their benefit functions from the project are *weakly increasing* in the level of the project. The weakly increasing benefit functions are worth analyzing because they are a generalization of the benefit functions of Bagnoli and Lipman (1989) and Branzei et al. (2005). We show that the unit-by-unit contribution mechanism always has a Nash equilibrium at which the nonharmful public project is undertaken Pareto-efficiently, although it may have a Nash equilibrium at which the project is done inefficiently. We further prove that with and

[4] For instance, Tokyo and Nagano city (a city about 220 km away from Tokyo) were connected by the HSR network in 1997. This network was extended to Kanazawa city (a city about 450 km away from Tokyo) in 2015.

[5] For a Japanese case, see, for example, Sasaki et al. (1997). Similar effects have been observed from the extension of HSR networks in European countries. See, for example, Ureña et al. (2009).

without monetary transfers, the set of Nash equilibria with efficient project implementation coincides with the set of strong Nash equilibria and the set of coalition-proof Nash equilibria (Bernheim et al., 1987) (Theorem 1). These results show that although multiple public project levels may be supported at the Nash equilibria, only Nash equilibria with efficient project implementation are supported by various Nash equilibrium refinements that are robust to coalition deviations. Theorem 1 supplements the results of earlier studies as follows: First, in the earlier studies, the weakly increasing property of the benefit functions is a key factor in the mechanism of efficient public good provision at a Nash equilibrium. Second, the Nash equilibria for efficient projects are much more robust to coalition deviations than are shown by Branzei et al. (2005) because they test only a strong Nash equilibrium without transfers.

Second, a project is considered *harmful for some agents* if their benefit functions from the project are not weakly increasing in its level. We additionally impose *weak concavity* on the benefit functions of all agents for tractability. We show that the unit-by-unit contribution mechanism does not always work well in the implementation of a harmful project. Unlike in nonharmful projects, this mechanism does not always have a Nash equilibrium with efficient public project implementation. Moreover, this mechanism may have a Nash equilibrium at which the project is undertaken over the efficient level. We establish necessary and sufficient conditions for a Nash equilibrium with implementation of the project at or over the efficient level (see Propositions 1 and 2). As for nonharmful projects, these conditions lead to the possibility of multiple Nash equilibria with both efficient and inefficient implementation of the public project. Then, we examine the strong Nash equilibrium and coalition-proof Nash equilibrium to clarify the level of project implementation—the efficient level or the over-implementation level—that is robust to coalition deviations. We observe that these refined Nash equilibria do not always select Nash equilibrium with efficient project implementation. First, we find that the mechanism may not have strong Nash equilibria with or without transfers. Second, although coalition-proof Nash equilibria with and without transfers do exist, they do not always single out Nash equilibria with efficient project implementation. Coalition-proof Nash equilibria single out Nash equilibria with efficient project implementation if and only if there is a Nash equilibrium with efficient project implementation, and no other Nash equilibria with over-implementation (Theorem 3). Finally, we introduce a reasonably large class of modified unit-by-unit contribution mechanism and investigate whether this modified mechanism achieves the efficient undertaking of harmful public projects. We show that no mechanism in this class implements the efficiency project in Nash equilibria (Proposition 5).

In conclusion, when the project is nonharmful for all agents, the unit-by-unit contribution mechanism works well since it only achieves an efficient project at various refined Nash equilibria. On the other hand, when the project is harmful for some agents, the mechanism does not necessarily work since it may not have a Nash equilibrium with an efficient project. Furthermore, even if it has such a Nash equilibrium, none of the refined Nash equilibrium based on coalition deviations considered in this paper singles it out. Thus, whether the unit-by-unit contribution mechanism works depends on the property of the project. The merit of the unit-by-unit contribution mechanism reported in the literature is extensible to the implementation of a nonharmful project, but only partially extensible to that of a harmful public project. If we aim to achieve efficient project implementation under general benefit structures at various refined Nash equilibria based on coalition deviations, we need to consider another class of modified unit-by-unit contribution mechanisms or construct new mechanisms.

Finally, we mention some related studies. Our conditions on benefit functions from a public project could be compared with several classes of benefit functions of Laussel and Le Breton (2001). In our model, if all agents have weakly increasing benefit functions, then the *comonotonicity* condition of Laussel and Le Breton (2001) holds. Otherwise, it does not. *The two-sided* property of Laussel and Le Breton (2001), another condition of benefit structures, does not hold in our model.[6] Thus, our benefit function conditions cannot be fully captured by the Laussel and Le Breton (2001) classes of benefit functions. In this sense, we analyze a new class of benefit functions. However, note that Laussel and Le Breton (2001) work on the common agency game, which is different from our unit-by-unit contribution game because ours does not have a profit-maximizing common agency to implement public projects. It seems less significant to compare their results with ours.

To the best of my knowledge, apart from Bagnoli and Lipman (1989) and Branzei et al. (2005), only Yu (2005) proposes a mechanism, which is completely different from the unit-by-unit contribution mechanism, for provision of an integer-unit pure public good. Her two-stage mechanism implements any one of the allocations in the core in an undominated subgame-perfect Nash equilibrium. A *voluntary participation problem*, pointed out by Saijo and Yamato (1999), can be captured as another free-rider problem of public good provision related to the participation decision in a public good mechanism. Nishimura and Shinohara (2013) propose a multi-stage mechanism, called a unit-by-unit *participation* mechanism, and show that the idea of a unit-by-unit process can mitigate this problem. Although the unit-by-unit participation mechanism and our mechanism are totally different, Nishimura and Shinohara (2013) do not explore the extensibility of the merit of the unit-by-unit participation mechanism to the implementation of harmful or nonharmful projects. Shinohara (2014) investigates a voluntary participation problem in which agents have the same benefit functions as those of Branzei et al. (2005). Shinohara (2014) does not study this extensibility, either.

The paper is organized as follows: Section 2 introduces the model and equilibrium concepts. Section 3 presents the results for nonharmful projects. Section 4 provides the results for harmful projects. Section 5 concludes the study. The proofs of the propositions in Sections 3 and 4 are collated in the appendices.

2 The model

Consider an economy in which agents undertake a public project through contribution of a private good (money). The level of the public project is assumed to take a nonnegative integer. Let $\mathcal{Y} = \{0, 1, \ldots, \bar{y}\}$ be the set of project levels, where $\bar{y}$ is an integer greater than or equal to one, and the finite upper bound of the public project level. Let $c : \mathcal{Y} \to \mathbb{R}_+$ be a cost function of the project such that $c(0) = 0$. For all $y, y' \in \mathcal{Y}$ such that $y \geq y'$, let $\Delta c(y, y') \equiv c(y) - c(y')$ be the additional (marginal) cost from y' to y

[6] For the definitions of comonotonicity and two-sidedness, see Laussel and Le Breton (2001).

units. We assume that c is an increasing and weakly convex function in $\mathcal{Y}$: that is,

$$\begin{aligned} &\Delta c(y+1, y) > 0 \text{ for all } y \in \mathcal{Y} \\ &\text{and } \Delta c(y+1, y) \geq \Delta c(y'+1, y') \text{ for all } y, y' \in \mathcal{Y} \text{ such that } y > y'. \end{aligned} \tag{1}$$

Let $N = \{1, \ldots, n\}$ be the set of agents such that n is a finite integer and $n \geq 1$. Each agent $i \in N$ has a quasi-linear utility function $U_i : \mathcal{Y} \times \mathbb{R}_+ \to \mathbb{R}$ such that $U_i(y, t_i) = u_i(y) - t_i$, in which $u_i : \mathcal{Y} \to \mathbb{R}$ is agent i's benefit function from the project with $u_i(0) = 0$ and t_i is i's private-good contribution to the project. For all $y, y' \in \mathcal{Y}$ such that $y \geq y'$, let $\Delta u_i(y, y') \equiv u_i(y) - u_i(y')$ be agent i's additional (marginal) benefit from the increase from y' to y units.

We assume that the project has a "public-good nature"; that is, every agent benefits from the same project level, irrespective of his contribution. However, we do not always assume that the project is a public "good." We allow the case in which a higher project level may harm some agents, while it benefits others. In the subsequent sections, we impose additional conditions on u_i, which determine the project character. Note that in our model, agents who benefit from a higher project level, if any, want to free-ride others' contribution. That is, the free-rider problem does matter.

We identify an *economy* by a list $[N, (u_i)_{i \in N}, c]$. For each economy, the existence of the Pareto-efficient level for a project is trivial since $\mathcal{Y}$ is a finite set. For analytical simplicity, we assume that $y^* \in \mathcal{Y}$ is a unique efficient project level, where y^* is positive; that is, $\{y^*\} = \arg\max_{y \in \mathcal{Y}} \sum_{i \in N} u_i(y) - c(y)$.[7] We also assume that for all coalitions $D \subseteq N$, $\arg\max_{y \in \mathcal{Y}} \sum_{j \in D} u_j(y) - c(y)$ is a singleton. For all $D \subseteq N$, let $Y(D) \in \mathcal{Y}$ be a *stand-alone level of the project for* D such that $\{Y(D)\} = \arg\max_{y \in \mathcal{Y}} \sum_{j \in D} u_j(y) - c(y)$. We do not assume that $Y(D)$ is positive for all $D \subsetneq N$. Let $Y^{\max} \equiv \max_{D \subseteq N} Y(D)$. The assumption of a unique stand-alone level for each coalition is used only in Section 4.

We immediately obtain Lemma 1 from the uniqueness of the efficient level y^*.

Lemma 1 For all $y \in \mathcal{Y}$, $\sum_{j \in N} \Delta u_j(y^*, y) > \Delta c(y^*, y)$ if $y^* > y$ and $\sum_{j \in N} \Delta u_j(y, y^*) < \Delta c(y, y^*)$ if $y^* < y$.

Proof. By the efficiency and the uniqueness of y^*, $\sum_{j \in N} u_j(y^*) - c(y^*) > \sum_{j \in N} u_j(y) - c(y)$ for all $y \in \mathcal{Y} \setminus \{y^*\}$, which implies the conditions in the statement. ■

We focus on the undertaking of a public project through a *unit-by-unit contribution mechanism*, which is the same as the mechanism of Branzei et al. (2005). In this mechanism, each agent $i \in N$ simultaneously chooses a vector of marginal contributions to each one-unit increase of the project. Let $\sigma_i \equiv (\sigma_i^y)_{y \in \mathcal{Y} \setminus \{0\}} \in \mathbb{R}_+^{\bar{y}}$ be a typical vector of marginal contributions chosen by agent i, in which $\sigma_i^y \in \mathbb{R}_+$ is a marginal contribution from i to the marginal production from $y - 1$ to y units. The project level is determined as follows: $y \in \mathcal{Y} \setminus \{0\}$ units of the project are undertaken at $\sigma = (\sigma_i)_{i \in N}$ if and only if (i) for all units of $\hat{y}$, which is less than or equal to y, the sum of contributions to the $\hat{y}$-th unit of the project, $\sum_{i \in N} \sigma_i^{\hat{y}}$, covers the marginal cost of that unit, $\Delta c(\hat{y}, \hat{y} - 1)$, and (ii) the sum of

[7] The subsequent analysis is applicable to the trivial case of $y^* = 0$.

contributions to $y+1$-th unit, $\sum_{i \in N} \sigma_i^{y+1}$, falls short of the marginal cost $\Delta c(y+1, y)$. If the marginal cost of the first unit is not covered by the sum of contributions to that unit, then the project level is zero. Formally, for each $\sigma = (\sigma_i)_{i \in N} \in \mathbb{R}_+^{n\bar{y}}$, let $y(\sigma)$ be the public project level at σ such that

$$y(\sigma) \equiv \max \left\{ y \in \mathcal{Y} \;\middle|\; \sum_{i \in N} \sigma_i^{\hat{y}} \geq \Delta c(\hat{y}, \hat{y}-1) \text{ for all } \hat{y} \in \mathcal{Y} \text{ such that } \hat{y} \leq y. \right\}, \tag{2}$$

where we define $\sigma_i^0 \equiv 0$ for all $i \in N$ and $c(0) - c(0-1) \equiv 0$ for consistency. For all $\sigma \in \mathbb{R}_+^{n\bar{y}}$, each agent i pays $\sum_{y \in \mathcal{Y} \setminus \{0\}} \sigma_i^y$. In this mechanism, the marginal contribution to some unit is never refunded even though the project is not undertaken at that unit. However, as we will see later, the contribution is never wasted at every Nash equilibrium.

The mechanism accompanied with $(U_i)_{i \in N}$ constitutes a strategic-form game $\Gamma = [N, (S_i, V_i)_{i \in N}]$, in which $S_i \equiv \mathbb{R}_+^{\bar{y}}$ is the set of strategies for $i \in N$ and $V_i : \prod_{j \in N} S_j \rightarrow \mathbb{R}$ is agent i's payoff function, depending on strategies such that $\sigma \in \prod_{j \in N} S_j \longmapsto V_i(\sigma) \equiv U_i(y(\sigma), \sum_{y \in \mathcal{Y} \setminus \{0\}} \sigma_i^y) \in \mathbb{R}$. Hereafter, we call Γ a *unit-by-unit contribution game*. The unit-by-unit contribution game is a complete information game.

Bagnoli and Lipman (1989) introduce a *multi-stage* unit-by-unit contribution mechanism. It starts with the decision on whether to provide the first unit of the project. In the first stage, the agents contribute to the first unit of the project. If the sum of contributions to the first unit covers the marginal cost for that unit, the first unit is provided, and the agents go to the second stage. Otherwise, the first unit is not provided, and the mechanism ends. If the agents go to the second stage, it is decided in the same way whether to provide a second unit. The second unit is provided, and the agents go to the third stage if and only if the sum of contributions to the second unit covers the marginal cost for that unit. This continues till the sum of contributions to a one-unit increase falls short of the marginal cost for that increase. We consider the mechanisms of Branzei et al. (2005) and Bagnoli and Lipman (1989) as essentially the same because the decision on a one-unit increase of the public good is based on the relationship between the marginal contribution and the marginal cost for that unit. In this paper, we analyze the mechanism based on a simultaneous game.

We introduce equilibrium concepts for the unit-by-unit contribution game. Our analysis is restricted to pure strategies. The Nash equilibrium is defined as usual.

For each $D \subseteq N$, denote a strategy profile for D by $\sigma_D \in \prod_{j \in D} S_j$. We simply write $\sigma_N = \sigma$. A *strong Nash equilibrium* (Aumann, 1959) is a Nash equilibrium that is stable against all possible coalition deviations.

Definition 1 Strategy profile $\sigma \in \prod_{j \in N} S_j$ is a *strong Nash equilibrium* of Γ if there is no $D \subseteq N$ and $\sigma'_D \in \prod_{j \in D} S_j$ such that $V_j(\sigma) < V_j(\sigma'_D, \sigma_{N \setminus D})$ for all $j \in D$.

A *coalition-proof Nash equilibrium* (Bernheim et al., 1987) is also an equilibrium based on stability against coordinated strategies. Unlike the strong Nash equilibrium, the coalition-proof Nash equilibrium is limited to "self-enforcing" coalitional deviations. This equilibrium is based on the notion of a restricted game. For all $D \subsetneq N$ and all $\sigma_{N \setminus D} \in \prod_{j \in N \setminus D} S_j$, $\Gamma | \sigma_{N \setminus D}$ is a *restricted game of* Γ *at*

$(D, \sigma_{N\setminus D})$ in which the agents in D plays Γ, taking as given that the other agents choose $\sigma_{N\setminus D}$; that is, $\Gamma|\sigma_{N\setminus D}$ is a list $[D, (S_i, \tilde{V}_i)_{i\in D}]$ in which D is a set of players for each $i \in D$, $S_i = \mathbb{R}_+^{\bar{y}}$ is i's strategy set, and $\tilde{V}_i$ is the payoff function of i such that $\tilde{\sigma}_D \in \prod_{i\in D} S_j \longmapsto \tilde{V}_i(\tilde{\sigma}_D) \equiv V_i(\tilde{\sigma}_D, \sigma_{N\setminus D}) \in \mathbb{R}$.

Definition 2 A coalition-proof Nash equilibrium $\sigma \in \prod_{j\in N} S_j$ is defined inductively with respect to the number of agents $n \geq 1$. Suppose that $n = 1$. Then, $\sigma \in \prod_{j\in N} S_j$ is a *coalition-proof Nash equilibrium of* Γ if σ is a Nash equilibrium of Γ.

Suppose that $n \geq 2$ and suppose that a coalition-proof Nash equilibrium has been defined for all games with fewer than n agents. $\sigma \in \prod_{j\in N} S_j$ is *self-enforcing in* Γ if it is a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N\setminus D}$ for all nonempty $D \subsetneq N$. $\sigma \in \prod_{j\in N} S_j$ is a *coalition-proof Nash equilibrium of* Γ if it is self-enforcing in Γ and there is no other self-enforcing strategies $\sigma' \in \prod_{j\in N} S_j$ in Γ such that $V_j(\sigma) < V_j(\sigma')$ for all $j \in N$.

The self-enforcing property of coalition-proof Nash equilibria restricts possible coalition deviations, and hence the set of strong Nash equilibria is always a subset of the set of coalition-proof Nash equilibria.

Since we assume that agents have quasi-linear utility functions, it would be appropriate to consider coalition deviations through monetary transfers. Consider a situation in which a coalition $D \subseteq N$ deviates and each of its members freely sends transfer to other members. Let $i \in D$ and $\tau_i \in \mathbb{R}$ be a *net* transfer to agent i from the others: τ_i is equal to the transfers i sends minus the transfers she receives. There is no outside transfer resource; that is, $\sum_{i\in D} \tau_i = 0$. Based on this kind of transfers, we redefine the strong Nash and coalition-proof Nash equilibria.

Definition 3 Strategy profile $\sigma \in \prod_{j\in N} S_j$ is a *strong Nash equilibrium with transfers* of Γ if there is no $D \subseteq N$, $\sigma'_D \in \prod_{j\in D} S_j$ and $(\tau_j)_{j\in D} \in \mathbb{R}^{|D|}$ such that $\sum_{j\in D} \tau_j = 0$ and $V_i(\sigma) < V_i(\sigma'_D, \sigma_{N\setminus D}) + \tau_i$ for all $i \in D$.

Note that σ is a strong Nash equilibrium with transfers if and only if there is no $D \subseteq N$ and $\sigma'_D \in \prod_{j\in D} S_j$ such that $\sum_{j\in D} V_j(\sigma) < \sum_{j\in D} V_j(\sigma'_D, \sigma_{N\setminus D})$. That is, no coalition can deviate from a strong Nash equilibrium with transfers so as to increase the sum of payoffs of its members.

Definition 4 A coalition-proof Nash equilibrium with transfers $\sigma \in \prod_{j\in N} S_j$ is defined inductively with respect to the number of agents $n \geq 1$. Suppose that $n = 1$. Then, $\sigma \in \prod_{j\in N} S_j$ is a *coalition-proof Nash equilibrium with transfers of* Γ if σ is a Nash equilibrium of Γ.

Suppose that $n \geq 2$ and suppose that a coalition-proof Nash equilibrium with transfers has been defined for all games with fewer than n agents. $\sigma \in \prod_{j\in N} S_j$ is *self-enforcing with transfers in* Γ if it is a coalition-proof Nash equilibrium with transfers of $\Gamma|\sigma_{N\setminus D}$ for all nonempty $D \subsetneq N$. $\sigma \in \prod_{j\in N} S_j$ is a *coalition-proof Nash equilibrium with transfers of* Γ if it is self-enforcing with transfers in Γ and there are no other self-enforcing strategies with transfers $\sigma' \in \prod_{j\in N} S_j$ in Γ and $(\tau_j)_{j\in N} \in \mathbb{R}^n$ such that $\sum_{j\in N} \tau_j = 0$ and $V_i(\sigma) < V_i(\sigma'_D, \sigma_{N\setminus D}) + \tau_i$ for all $i \in N$.

Note that $\sigma \in \prod_{j\in N} S_j$ is a coalition-proof Nash equilibrium with transfers of Γ if and only if it is self-enforcing with transfers in Γ and there are no self-enforcing strategies with transfers $\sigma' \in \prod_{j\in N} S_j$ such that $\sum_{j\in N} V_j(\sigma) < \sum_{j\in N} V_j(\sigma')$.

About the strong Nash equilibrium, since monetary transfers increase the possibility of coalition deviations, every strong Nash equilibrium with transfers is generally a strong Nash equilibrium, but the converse is not necessarily true. However, the same does not apply to a coalition-proof Nash equilibrium. The two sets of coalition-proof Nash equilibria may be disjoint. See Appendix C.

Remark 1 The remarks on the above equilibria are in order. (i) Every strong Nash equilibrium with transfers is a strong Nash equilibrium, which in turn is a coalition-proof Nash equilibrium. (ii) Every strong Nash equilibrium with transfers is a coalition-proof Nash equilibrium with transfers. (iii) In Γ, no coalition-proof Nash equilibrium is Pareto-dominated by other coalition-proof Nash equilibria. (iv) There are never two distinct coalition-proof Nash equilibria with transfers that take different values of the sum of the payoffs to agents.

3 Results: Nonharmful public projects

We consider an economy in which agents undertake a project that is *nonharmful* for all agents in the sense that the increase in project level does not harm any agent. This economy is formally defined as a list $[N, (u_i)_{i\in N}, c]$ in which u_j is *weakly increasing* in the project level for all $j \in N$: for all $j \in N$ and all $y \in \mathcal{Y}$,

$$\Delta u_j(y+1, y) \geq 0 \tag{3}$$

and c is weakly convex and increasing in the level (see (1)). We refer to this economy as $\mathbf{e^1}$.

Theorem 1 For an economy $\mathbf{e^1} = [N, (u_i)_{i\in N}, c]$, in the unit-by-unit contribution game, (i) there is no Nash equilibrium at which the project is undertaken over level y^* and (ii) the set of Nash equilibria at which the project is undertaken at level y^* coincides with the sets of strong Nash equilibria with and without transfers and the sets of coalition-proof Nash equilibria with and without transfers, and all sets are nonempty.

The proof is provided in the appendix. The project levels at Nash equilibria may be multiple, but at most y^*.[8] Since strong Nash equilibria and coalition-proof Nash equilibria single out Nash equilibria with efficient project implementation levels, coordination possibilities modeled through those equilibria successfully lead to efficient allocation. In this sense, given coordination possibilities, the unit-by-unit contribution mechanism is successful in the implementation of nonharmful projects.

Studies on the provision of integer-unit public goods have examined several distinct benefit functions. Bagnoli and Lipmann (1989) and Nishimura and Shinohara (2013) assume that agents' benefit functions are strictly increasing in the public good level. Moreover, Bagnoli and Lipmann (1989) impose strict

[8] We can make an example in which the unit-by-unit contribution game may have Nash equilibria at which the project is undertaken below y^*. For example, consider a case of $\mathcal{Y} = \{0, 1, 2\}$, $c(y) = 10y$ for all $y \in \mathcal{Y}$, $N = \{1, 2\}$, and $u_i(1) = 7$ and $u_2(2) = 13$ for all $i \in N$.

concavity on the benefit functions. Branzei et al. (2005) and Shinohara (2014) assume that every agent $i \in N$ has a discontinuous benefit function such that there is a threshold level of the public good y_i and a positive constant value $\bar{u}_i$ such that $u_i(y) = \bar{u}_i$ if $y \geq y_i$ and $u_i(y) = 0$ otherwise. Obviously, all of the benefit functions in the literature are examples of weakly increasing benefit functions. The existence of Nash equilibria with efficient projects, shown by Bagnoli and Lipmann (1989) and Branzei et al. (2005), is extensible to the case in which agents have weakly increasing benefit functions.

By Theorem 1, we observe that the Nash equilibrium with an efficient project is robust to several types of coalitional deviations. This robustness property is stronger than the finding by Branzei et al. (2005). This is because while Branzei et al. (2005) examine a strong Nash equilibrium (without transfers), we examine four refined Nash equilibria, including a strong Nash equilibrium.[9]

4 Results: Harmful public projects

To what extent are the desirable properties of the unit-by-unit contribution mechanism, shown in Theorem 1, satisfied when implementing a public project that is sometimes *harmful* to some agents? We consider an economy in which at least one agent has a benefit function that is not weakly increasing, that is, an economy $[N, (u_i)_{i \in N}, c]$ in which there exist $j \in N$ and $y_j \in \mathcal{Y} \backslash \{\bar{y}\}$ such that $\Delta u_j(y_j + 1, y_j) < 0$ and c satisfies (1). In this economy, some agents such as agent j above do not always benefit from an increase in the project level.

First, we provide examples to show that in this economy, the unit-by-unit contribution mechanism may not achieve an efficient project level at some refined Nash equilibria.

Example 1 Let $\mathcal{Y} = \{0, 1, 2\}$. Let $c(y) = 10y$ for all $y \in \mathcal{Y}$. Let $N = \{1, 2\}$. Suppose $u_1(1) = 4$, $u_1(2) = 1$, $u_2(1) = 12$, and $u_2(2) = 23$. Then, $y^* = 1$ and $Y^{\max} = 2$. First, we show that no Nash equilibrium supports the efficient undertaking of the project. Take $\sigma = (\sigma_1^1, \sigma_1^2; \sigma_2^1, \sigma_2^2)$ such that $\sigma_1^1 + \sigma_2^1 = 10$ and $\sigma_1^2 + \sigma_2^2 < 10$. In this σ, $y(\sigma) = 1$. However, it cannot be a Nash equilibrium because if agent 2 increases his marginal contribution to the second unit from σ_2^2 to $10 - \sigma_1^2$, then he is made better off (note that $\Delta u_2(2, 1) > \Delta c(2, 1) \geq \Delta c(2, 1) - \sigma_1^2$ in this example). We can easily verify that $\sigma' \in \prod_{j \in N} S_j$ such that $\sigma_1' = (0, 0)$ and $\sigma_2' = (10, 10)$ is a unique Nash equilibrium that is also coalition-proof. Second, we can verify that no strong Nash equilibrium exists since σ' is not a strong Nash equilibrium with or without transfers (consider a deviation by N from σ' to $\tilde{\sigma} \in \prod_{j \in N} S_j$ such that $(\tilde{\sigma}_1^1, \tilde{\sigma}_1^2) = (2, 0)$ and $(\tilde{\sigma}_2^1, \tilde{\sigma}_2^2) = (8, 0)$).

Example 2 Let $\mathcal{Y} = \{0, 1, 2\}$ and $c(y) = 10y$. Let $N = \{1, 2, 3, 4\}$. Suppose that $u_1(1) = 7.5$ and $u_1(2) = 0$ and that $u_i(1) = 6$ and $u_i(2) = 12$ for all $i \in N \backslash \{1\}$. Then, $y^* = Y^{\max} = 2$. In this example, we show that there is no strong Nash equilibrium with transfers at which the project is undertaken at level y^*, while there exists a strong Nash equilibrium.

[9] In regard to the result in Branzei et al. (2005), it would be important to discuss whether a Nash equilibrium with an efficient project achieves the core of some cooperative game. This is because Branzei et al. (2005) show that utility allocations attained at strong Nash equilibria are the core of a cooperative game. We can show that if agents have weakly increasing benefit functions, all utility allocations at the strong Nash equilibria belong to the core of a cooperative game. The proof is available upon request.

We can find a strategy profile that is a strong Nash equilibria. For example, consider $\sigma \in \prod_{j \in N} S_j$ such that $(\sigma_1^1, \sigma_1^1) = (0,0)$, $(\sigma_2^1, \sigma_2^1) = (10,0)$, and $(\sigma_i^1, \sigma_i^1) = (0,5)$ for $i = 3,4$, which is a strong Nash equilibrium.

Second, we show that there exists no strong Nash equilibrium with transfers. Let σ be a Nash equilibrium such that $y(\sigma) = 2$. Since $u_1(2) = 0$ and σ is a Nash equilibrium, we obtain $(\sigma_1^1, \sigma_1^2) = (0,0)$. We further obtain $\sum_{i \in N\backslash\{1\}} \sigma_i^y = \Delta c(y, y-1)$ for all $y \in \mathcal{Y}\backslash\{0\}$ (see Lemma A1 in Appendix A). At σ, $V_1(\sigma) = 0$ and $V_i(\sigma) = 12 - \sigma_i^1 - \sigma_i^2$ for all $i \in N\backslash\{1\}$. Now, we consider a coalition $\{1, j\}$ such that $j \in N\backslash\{1\}$ and $\sigma_j^2 > 0$. Suppose that this coalition deviates from σ to $\tilde{\sigma}_{\{1,j\}}$ such that $\tilde{\sigma}_1 = \sigma_1$ and $\tilde{\sigma}_j = (\sigma_j^1, 0)$. Then, $y(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}})$=1 and $V_1(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) + V_j(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) = 7.5 + 6 - \sigma_j^1$. Finally,

$$V_1(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) + V_j(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) - \Big(V_1(\sigma) + V_j(\sigma)\Big) = 1.5 + \sigma_j^2 > 0.$$

Thus, no strong Nash equilibrium with transfers exists.

In those examples, there is only one agent whose benefit function is not weakly increasing. Nevertheless, the equilibria of the unit-by-unit contribution game have properties that are very different from those in Theorem 1. First, a Nash equilibrium may not support the efficient project y^* (see Example 1). Second, strong Nash equilibria with and without transfers may not exist. Moreover, no strong Nash equilibrium with transfers may exist in either $Y^{\max} = y^*$ or $Y^{\max} > y^*$. Third, although coalition-proof Nash equilibria with and without transfers exist in those examples, they do not always support an efficient project.

By Examples 1 and 2, the unit-by-unit contribution mechanism does not necessarily achieve an efficient project at refined Nash equilibria, unlike in the implementation of nonharmful projects. In particular, it is impossible for the mechanism to achieve efficiency through a strong Nash equilibrium since it may not exist. We now focus on the coalition-proof Nash equilibria and examine to what extent the unit-by-unit contribution mechanism achieves an efficient project level in an economy with harmful projects.

The condition that at least one agent does not have a weakly increasing benefit function seems very weak, and hence we need to consider many economies for the analysis. For tractability, we focus on a subclass of such economies, in which agents have *weakly concave* benefit functions. Formally, we consider an economy $\mathbf{e^2} = [N, (u_i)_{i \in N}, c]$ in which some agents do not have weakly increasing benefit functions; that is, there exist $j \in N$ and $y_j \in \mathcal{Y}\backslash\{\bar{y}\}$ such that

$$\Delta u_j(y_j + 1, y_j) < 0, \tag{4}$$

every agent has weakly concave benefit functions: for all $i \in N$ and all $y, y' \in \mathcal{Y}$ such that $\bar{y} > y > y'$,

$$\Delta u_i(y + 1, y) \leq \Delta u_i(y' + 1, y'), \tag{5}$$

and c is weakly convex and increasing (see (1)).

Note that there may be agents whose benefit functions are weakly increasing.

Lemma 2 In economy $\mathbf{e^2}$, for all $j \in N$, if u_j satisfies (4), then there exists $p(j) \in \mathcal{Y} \backslash \{\bar{y}\}$ such that

$$\begin{aligned} &\Delta u_i(y+1, y) \geq 0 \text{ for all } y \in \mathcal{Y} \text{ such that } y < p(j) \\ \text{and } &\Delta u_i(y+1, y) < 0 \text{ for all } y \in \mathcal{Y} \text{ such that } y \geq p(j). \end{aligned} \tag{6}$$

Proof. Suppose that u_j satisfies (4). By a weak concavity of u_j, $\Delta u_j(y+1, y) < 0$ for all $y \in \mathcal{Y} \backslash \{\bar{y}\}$ such that $y \geq y_j$. Thus, if there exists $y' \in \mathcal{Y} \backslash \{0, \bar{y}\}$ such that $\Delta u_j(y', y'-1) \geq 0$ and $\Delta u_j(y'+1, y') < 0$, then we can define $y' \equiv p(j)$. Otherwise, $p(j) \equiv 0$. ■

Our interpretation is that $p(j)$ is the level of the public project that peaks agent j's benefit from the project. In this economy, every agent whose benefit function is not weakly increasing has a *peaked benefit function*. For convenience, we also define this peak level of the project for all agents whose benefit function is weakly increasing as follows: $p(i) \equiv \bar{y}$ for all $i \in N$ such that $\Delta u_i(y+1, y) \geq 0$ for all $y \in \mathcal{Y} \backslash \{\bar{y}\}$. We further introduce some notations for the analysis. For all $y \in \mathcal{Y}$, let $N^y \equiv \{i \in N | \, p(i) \geq y\}$: the set of agents whose peak level is not less than y. Then, for all $i \in N$ and $y \in \mathcal{Y} \backslash \{0\}$, $i \in N^y$ if and only if $\Delta u_i(y, y-1) \geq 0$.

In this economy, although y^* is a unique efficient level of the project, $Y^{\max}$ may not be equal to y^*. We provide a necessary and sufficient condition under which $Y^{\max} = y^*$ in Lemma 3, which would be useful for subsequent analyses.

Lemma 3 In economy $\mathbf{e^2}$, $Y^{\max} = y^*$ if and only if

$$\sum_{j \in N^{y^*+1}} \Delta u_j(y^*+1, y^*) < \Delta c(y^*+1, y^*). \tag{7}$$

Proof. ($\Leftarrow$) By the definition of $Y^{\max}$, $y^* \leq Y^{\max}$. Suppose, to the contrary, that $y^*+1 \leq Y^{\max}$. Let $D \subseteq N$ be such that $Y(D) = Y^{\max}$. Then, $\sum_{j \in D} \Delta u_j(Y(D), y^*) > \Delta c(Y(D), y^*)$.

Note that $\sum_{j \in D} \Delta u_j(y^*+1, y^*) \leq \sum_{j \in N^{y^*+1}} \Delta u_j(y^*+1, y^*)$. This inequality, together with (1), (5), and (7), implies that for all $y \in \mathcal{Y}$ such that $y \geq y^*+1$,

$$\sum_{j \in D} \Delta u_j(y+1, y) \leq \sum_{j \in D} \Delta u_j(y^*+1, y^*) < \Delta c(y^*+1, y^*) \leq \Delta c(y+1, y).$$

Finally, we obtain $\sum_{j \in D} \Delta u_j(Y(D), y^*) < \Delta c(Y(D), y^*)$, which is a contradiction.

($\Rightarrow$) $Y^{\max} = y^*$ implies that $Y(N^{y^*+1}) \leq y^*$. Since $Y(N^{y^*+1})$ is the unique maximizer of $\sum_{j \in N^{y^*+1}} u_j(y) - c(y)$,

$$\sum_{j \in N^{y^*+1}} \Delta u_j(Y(N^{y^*+1})+1, Y(N^{y^*+1})) < \Delta c(Y(N^{y^*+1})+1, Y(N^{y^*+1})).$$

By (1), (5), and $Y(N^{y^*+1}) \leq y^*$,

$$\sum_{j \in N^{y^*+1}} \Delta u_j(y^*+1, y^*) \leq \sum_{j \in N^{y^*+1}} \Delta u_j(Y(N^{y^*+1})+1, Y(N^{y^*+1}))$$

and

$$\Delta c(Y(N^{y^*+1})+1, Y(N^{y^*+1})) \leq \Delta c(y^*+1, y^*).$$

Thus, we obtain (7). ∎

Lemma 4 is a preparation for subsequent analyses.

Lemma 4 In economy $\mathbf{e^2}$, $\sum_{i\in N^{y^*}} \Delta u_i(y, y-1) > \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y^*$.

Proof. By Lemma 1, $\sum_{i\in N} \Delta u_i(y^*, y^*-1) > \Delta c(y^*, y^*-1)$. Since $\sum_{i\notin N^{y^*}} \Delta u_i(y^*, y^*-1) < 0$ (if $N^{y^*} \neq \emptyset$),

$$\sum_{i\in N} \Delta u_i(y^*, y^*-1) = \sum_{i\in N^{y^*}} \Delta u_i(y^*, y^*-1) + \sum_{i\notin N^{y^*}} \Delta u_i(y^*, y^*-1) \leq \sum_{i\in N^{y^*}} \Delta u_i(y^*, y^*-1).$$

Thus, $\sum_{i\in N^{y^*}} \Delta u_i(y^*, y^*-1) > \Delta c(y^*, y^*-1)$. By (1) and (5), we obtain the condition in the statement of this lemma. ∎

4.1 Nash equilibria and Pareto efficiency

As Examples 1 and 2 show, the unit-by-unit contribution game may or may not have a Nash equilibrium that undertakes the project efficiently in economy $\mathbf{e^2}$. Hence, we investigate under which conditions the unit-by-unit contribution game has such a Nash equilibrium in $\mathbf{e^2}$.

By Lemma 4, we can construct a strategy profile $\sigma^* \in \prod_{j\in N} S_j$ such that:

$$\begin{aligned}
&\text{If } y^*+1 \leq y \leq \bar{y}, \text{ then } \sigma_i^{*y} = 0 \text{ for all } i \in N.\\
&\text{If } 1 \leq y \leq y^*, \text{then } \sigma_i^{*y} = 0 \text{ for all } i \notin N^{y^*}, 0 \leq \sigma_i^{*y} < \Delta u_i(y, y-1) \text{ for all } i \in N^{y^*},\\
&\text{and } \sum_{i\in N^{y^*}} \sigma_i^{*y} = \Delta c(y, y-1).
\end{aligned} \tag{8}$$

At σ^*, the project is undertaken efficiently. However, as we can easily check, while it is a Nash equilibrium in Example 2, it is not in Example 1. Hence, some conditions are needed for it to be a Nash equilibrium. Profile σ^* plays an important role in establishing a necessary and sufficient condition for a Nash equilibrium to achieve the efficient project level.

Proposition 1 *In economy* $\mathbf{e^2}$*, the unit-by-unit contribution game has a Nash equilibrium at which the project is undertaken efficiently if and only if*

$$\Delta u_i(y^*+1, y^*) \leq \Delta c(y^*+1, y^*) \textit{ for all } i \in N. \tag{9}$$

The proof is provided in the appendix. Whether a Nash equilibrium with an efficient project level exists depends on the relationship between the marginal benefits of each agent and marginal cost for the y^*+1-th unit of the project. We can intuitively understand (9). If (9) holds, no agent gains from a one-unit increase in the project level from y^* units. Hence, without (9), no Nash equilibrium supports an efficient project level.

Corollary 1 establishes a sufficient condition under which the unit-by-unit contribution game has a Nash equilibrium with an efficient project.

Corollary 1 In economy $\mathbf{e^2}$, if $Y^{\max} = y^*$, then there is a Nash equilibrium at which the project is undertaken at the level y^*.

Proof. $Y^{\max} = y^*$ implies (7), which in turn implies (9). Thus, by Proposition 1, a Nash equilibrium exists such that the public project is provided efficiently. ■

It is easily seen that $Y^{\max} = y^*$ is not a necessary condition for a Nash equilibrium to support an efficient project.

As Example 1 shows, in economy $\mathbf{e^2}$, there may be a Nash equilibrium that supports over-implementation of the project in the unit-by-unit contribution game. Proposition 2 provides a necessary and sufficient condition for an economy under which the project is implemented over the efficient level at a Nash equilibrium.

Proposition 2 *In economy $\mathbf{e^2}$, there exists a Nash equilibrium at which the project is undertaken over the efficient level y^* if and only if $Y^{\max} > y^*$.*

The proof is provided in the appendix. We can intuitively interpret $Y^{\max} > y^*$. $Y^{\max} > y^*$ if and only if $Y(D) > y^*$ for some $D \subsetneq N$. Coalition D can undertake the public project at the level $Y(D)$ by itself since $\sum_{j \in D} \Delta u_j(y, y-1) \geq \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y(D)$. Thus, there exists a proper subgroup of N that can undertake the project over y^*. In conclusion, whether over-implementation of the project is supported at a Nash equilibrium depends on the existence of such a subgroup.

Theorem 2 summarizes the results in subsection 4.1, which is direct from Corollary 1 and Proposition 2.

Theorem 2 *In economy $\mathbf{e^2}$, (i) if $Y^{\max} = y^*$, then there exists a Nash equilibrium that supports the efficient project, but there is no Nash equilibrium at which the project is undertaken over the efficient level. (ii) If $Y^{\max} > y^*$, there exists a Nash equilibrium with over-implementation of the project, and there may be a Nash equilibrium that supports the efficient project.*

Remarks about Theorem 2 are in order. First, in the case of $Y^{\max} = y^*$, there may be a Nash equilibrium that supports underprovision of the public project. Second, in the case of $Y^{\max} > y^*$, there may be a Nash equilibrium with efficient project implementation. Thus, as in economy $\mathbf{e^1}$, the unit-by-unit contribution mechanism may face multiplicity of Nash equilibria that support different project levels in economy $\mathbf{e^2}$.

4.2 Coalition-proof Nash equilibria and Pareto efficiency

We examine which Nash equilibria are coalition-proof both with and without transfers in the unit-by-unit contribution game in economy $\mathbf{e^2}$. Lemma 5 is a preliminary for the analysis.

Lemma 5 There exists $\tilde{M} \subseteq N$ that satisfies

$$\Delta u_j(y, y-1) > 0 \text{ for all } j \in \tilde{M} \text{ and for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}, \tag{10}$$

$$\text{and} \sum_{j \in \tilde{M}} \Delta u_j(y, y-1) > \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}. \tag{11}$$

The proof is provided in the appendix. Let $\mathcal{M} \subseteq N$ be the "largest" set that satisfies (10) and (11) in the sense that no other sets satisfy these conditions or include $\mathcal{M}$. By these conditions, we can find $\sigma \in \mathbb{R}_+^{n\bar{y}}$ such that

$$\begin{aligned}
&\text{If } Y^{\max}+1 \leq y \leq \bar{y}, \text{ then } \sigma_i^y = 0 \text{ for all } i \in N.\\
&\text{If } 1 \leq y \leq Y^{\max}, \text{ then } \sigma_i^y = 0 \text{ for all } i \notin \mathcal{M},\ 0 < \sigma_i^y < \Delta u_i(y, y-1) \text{ for all } i \in \mathcal{M},\\
&\text{and} \sum_{j \in \mathcal{M}} \sigma_j^y = \Delta c(y, y-1).
\end{aligned} \tag{12}$$

By (12), for all $j \in \mathcal{M}$,

$$\Delta u_j(Y^{\max}, \hat{y}-1) > \sum_{y=\hat{y}}^{Y^{\max}} \sigma_j^y \text{ for all } \hat{y} \in \mathcal{Y} \text{ such that } 1 \leq \hat{y} \leq Y^{\max}. \tag{13}$$

In the proof of Proposition 3, provided in the appendix, we show that σ in (12) is a coalition-proof Nash equilibrium with and without transfers.

Proposition 3 *In economy* $\mathbf{e^2}$*, the unit-by-unit contribution game has coalition-proof Nash equilibria with and without transfers at which the project is undertaken at the level* $Y^{\max}$.

The proof is provided in the appendix. Although the unit-by-unit contribution game may not have strong Nash equilibria with or without transfers, it always has coalition-proof Nash equilibria with and without transfers. The self-enforcing property of coalition deviations guarantees the existence of coalition-proof Nash equilibria. It would be useful to again consider Example 1 in order to intuitively understand how this property works. Recall that $y^* = 1$ and $Y^{\max} = 2$ in this example. Recall also that $\sigma' = (\sigma_1'^1, \sigma_1'^2; \sigma_2'^1, \sigma_2'^2) = (0, 0; 10, 10)$ is the Nash equilibrium, but it is not a strong Nash equilibrium because N has a profitable deviation $\tilde{\sigma} = (\tilde{\sigma}_1^1, \tilde{\sigma}_1^2; \tilde{\sigma}_2^1, \tilde{\sigma}_2^2) = (2, 0; 8, 0)$ from σ'. By this deviation, the project level declines from $y(\sigma') = 2$ to $y(\tilde{\sigma}) = 1$. However, this deviation is not self-enforcing because agent 2 is willing to get the project level back to two units after the deviation. This is because agent 2's marginal benefit from the second unit is greater than the marginal cost for the unit. When the self-enforcing property matters, agents 1 and 2 do not agree with the first joint deviation to decrease the project level. In general, in the unit-by-unit contribution game in economy $\mathbf{e^2}$, the deviation to decrease the project level from $Y^{\max}$ is not self-enforcing (see the proof of Proposition 3 for details).

The next proposition shows that no coalition-proof Nash equilibrium supports the public project under $Y^{\max}$.

Proposition 4 *Suppose that there exists a Nash equilibrium $\hat{\sigma} \in \mathbb{R}_+^{n\bar{y}}$ such that $y(\hat{\sigma}) < Y^{\max}$ in the unit-by-unit contribution game in economy $\mathbf{e^2}$. Then, $\hat{\sigma}$ is not a coalition-proof Nash equilibrium with or without transfers.*

The proof is provided in the appendix. As Proposition 3 shows, there is a coalition-proof Nash equilibrium at which the project is undertaken at the level $Y^{\max}$. This, together with (13), implies that agents in $\mathcal{M}$ have a self-enforcing deviation from $\hat{\sigma}$ in a way that increases the project level from $y(\hat{\sigma})$ to $Y^{\max}$ and makes all of them better off (see the proof of Proposition 4 for details).

Theorem 3 summarizes under what condition a coalition-proof Nash equilibrium achieves the efficient project level.

Theorem 3 *In economy $\mathbf{e^2}$, (i) the project is undertaken at the level y^* at all coalition-proof Nash equilibria with and without transfers if $Y^{\max} = y^*$. (ii) The project is undertaken over the efficient level at all coalition-proof Nash equilibria if $Y^{\max} > y^*$.*

The proof is provided in the appendix. When the project is harmful for some agents, a Nash equilibrium itself may not support the efficient project. In some cases, multiple Nash equilibria support both efficient implementation and over-implementation of the project. In these cases, no Nash equilibrium with the efficient project is robust to coalition deviations. No strong Nash equilibria may exist. Coalition-proof Nash equilibria with and without transfers always exist, but they single out the Nash equilibrium with over-implementation of the project. These results differ greatly from those for nonharmful project implementation.

4.3 A modified mechanism

At the discussion after Proposition 2, we mention that $Y^{\max} > y^*$ means the existence of a group that can over-implement the project. If such a group exists, agents outsider this group cannot prevent the over-implementation of the project because they can announce only nonnegative contributions for each one-unit increase. Now, we modify the unit-by-unit contribution mechanism in such a way that agents can announce negative numbers for each one-unit increase. Let $S_i \equiv \mathbb{R}^{\bar{y}}$ for each $i \in N$. For each $\sigma \in \prod_{j \in N} S_j$, the level of the project $y(\sigma)$ is defined as the same as (2). For each $i \in N$ and $(\sigma_i^y)_{y \in \mathcal{Y} \setminus \{0\}} \in S_i$, $(t_i^y(\sigma_i^y))_{y \in \mathcal{Y} \setminus \{0\}}$ is defined as a vector of i's *actual* contributions for each one-unit increase such that for each $y \in Y \setminus \{0\}$, $t_i^y(\sigma_i^y) = \sigma_i^y$ if $\sigma_i^y \geq 0$ and $t_i^y(\sigma_i^y)$ takes a positive value otherwise. For each $\sigma \in \prod_{j \in N} S_j$ and each $i \in N$, $V_i(\sigma) = u_i(y(\sigma)) - \sum_{y \in Y \setminus \{0\}} t_i^y(\sigma_i^y)$. Note that by this modification, agents can prevent any one-unit increase if they announce sufficiently small negative numbers.

We can intuitively understand this modified mechanism that for each one-unit increase, each agent is asked to announce "willingness to pay" for carrying out the increase (a positive number) or that for preventing it (a negative number). Whether the project increases by one unit depends upon the sum of the announced willingness-to-pay. For any one-unit increase, if agents announce a positive value for that increase, they make the same payment as their announcement. Otherwise, their actual payment for that increase can be any positive value. There are some examples about how to set the actual contributions for negative numbers. For example, consider that $t_i^y(\sigma_i^y) = |\sigma_i^y|$ when $\sigma_i^y < 0$. In this example, agents who announce a negative number for some one-unit increase pay the absolute value of their willingness to pay for preventing that increase. We can consider another example in which for each $y \in \mathcal{Y} \backslash \{0\}$, $t_i^y(\sigma_i^y) = \varepsilon$ for some positive constant ε when $\sigma_i^y < 0$. We can intuitively understand that ε is a "fine" for announcing a negative number. It is enough that ε is very close to zero.

Proposition 5 *In some economy* $\mathbf{e^2}$*, no Nash equilibrium supports the efficient implement of the public project in the modified unit-by-unit contribution mechanism.*

The proof is provided in the appendix.

In this mechanism, if agents announce negative values for some one-unit increase, then their payment for that increase can take any positive value. In this sense, this modified mechanism seems to constitute a reasonably large class of modification of the unit-by-unit contribution mechanisms. By Proposition 5, we confirm that introducing negative contributions to the unit-by-unit contribution mechanism is not sufficient for the efficient undertaking of the project. We also confirm that if agents announce negative numbers for some one-unit increase, then they are subsidized to some extent, but not asked to contribute.

5 Conclusion

The unit-by-unit contribution mechanism seems suitable for the implementation of integer-unit public projects and applicable, to some extent, for public project initiatives in the real world. Hence, it is important that we understand how this mechanism works in the implementation of various public projects. However, this issue has received only limited attention. Our aim is to examine to what extent this mechanism achieves Pareto efficiency in the implementation of public projects. We consider not only a project that is nonharmful for all agents but also one that is not.

Our results are as follows. The mechanism works well in an economy in which the project is nonharmful for all agents. In this economy, the mechanism achieves an efficient project level only at a strong Nash equilibrium and a coalition-proof Nash equilibrium with and without transfers. In this sense, given various coalitional behaviors, the mechanism achieves efficiency. On the other hand, in other economies, the mechanism does not always work well. When the project is harmful for some agents, the unit-by-unit contribution mechanism does not necessarily have a Nash equilibrium with an efficient project. Even if the mechanism has such a Nash equilibrium, it is not necessarily supported at a strong Nash equilibrium or a coalition-proof Nash equilibrium. We introduce a reasonable class of modified unit-by-unit contribution mechanisms, but no mechanism in this class achieve implements efficient public project in Nash equilibria. We conclude that the unit-by-unit contribution mechanism should be used only for public projects that benefit all agents. In order to achieve an efficient project level that is harmful for some, we need to consider another class of modified unit-by-unit contribution mechanism or construct a completely new mechanism to undertake public projects. This is left for future research.

Appendix A: Preliminary results

In Appendix A, we examine a unit-by-unit contribution game without (1), (3), or (5). Instead of these conditions, we impose other conditions on the benefit and cost functions in each of subsequent lemmas. The results obtained in this appendix are applied to prove the results in the main text.

Let $\Gamma^0 = [\mathcal{N}, (S_i, \mathcal{V}_i)_{i\in\mathcal{N}}]$ be a unit-by-unit contribution game where $\mathcal{N}$ is the set of agents, S_i is i's set of strategies such that $S_i = \mathbb{R}_+^{\bar{y}}$, and $\mathcal{V}_i : \prod_{j\in\mathcal{N}} S_j \to \mathbb{R}$ is i's payoff function such that $\sigma_{\mathcal{N}} \in \prod_{j\in\mathcal{N}} S_j \longmapsto \mathcal{V}_i(\sigma_{\mathcal{N}}) \equiv u_i(y(\sigma_{\mathcal{N}})) - \sum_{y=1}^{\bar{y}} \sigma_i^y \in \mathbb{R}$ where $y : \prod_{j\in\mathcal{N}} S_j \to \mathcal{Y}$ is a mapping assigning a level of the public project with each strategy profile, which is defined in the same way as (2) in the main text. We assume that for all $y \in \mathcal{Y}$, $c(y) \geq 0$ and for all $y \in \mathcal{Y}\backslash\{0\}$, $\Delta c(y, y-1) \equiv \max\{0, c(y) - c(y-1)\}$. However, we do not impose any of (1), (3), and (5) on Γ^0.

A.1 Results of Nash equilibria of Γ^0

Lemma A1 shows that the contributions at every Nash equilibrium satisfy the budget balance condition.

Lemma A1 Suppose that $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$ is a Nash equilibrium of Γ^0. Then,

$$\sum_{j\in\mathcal{N}} \sigma_j^y = \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y(\sigma_{\mathcal{N}}) \text{ if } y(\sigma_{\mathcal{N}}) \geq 1. \tag{14}$$

$$\sigma_j^y = 0 \text{ for all } j \in \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq y(\sigma_{\mathcal{N}}) + 1 \text{ if } y(\sigma_{\mathcal{N}}) + 1 \leq \bar{y}. \tag{15}$$

Proof. *Proof of* (14). Since the project is undertaken at the level $y(\sigma_{\mathcal{N}})$ at $\sigma_{\mathcal{N}}$, $\sum_{j\in\mathcal{N}} \sigma_j^y \geq \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y(\sigma)$. Suppose, to the contrary, that there exists $\tilde{y} \in \mathcal{Y}$ such that $1 \leq \tilde{y} \leq y(\sigma)$ and $\sum_{j\in\mathcal{N}} \sigma_j^{\tilde{y}} > \Delta c(\tilde{y}, \tilde{y}-1)$. Then, clearly, there exists $i \in \mathcal{N}$ such that $\sigma_i^{\tilde{y}} > 0$. Even if this agent i decreases his contribution to $\tilde{y}$-th unit from $\sigma_i^{\tilde{y}}$ to $\sigma_i'^{\tilde{y}} = \max\left\{0,\ \Delta c(\tilde{y}, \tilde{y}-1) - \sum_{j\in\mathcal{N}\backslash\{i\}} \sigma_j^{\tilde{y}}\right\}$, he can still enjoy the project at the level $y(\tilde{\sigma}_{\mathcal{N}})$ while his total contribution decreases. Hence, he is made better off by this deviation, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium.

Proof of (15). Suppose that there exist $j \in N$ and $\tilde{y} \in \mathcal{Y}$ such that $\tilde{y} \geq y(\sigma_{\mathcal{N}}) + 1$ and $\sigma_j^{\tilde{y}} > 0$. If agent j switches from $\sigma_j^{\tilde{y}}$ to $\sigma_j'^{\tilde{y}} = 0$, the level of the project does not change. Hence, by this switch, agent j can still enjoy the project at the level $y(\sigma_{\mathcal{N}})$ as well as reduce his contribution, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium. ■

Lemma A2 proves that at every Nash equilibrium, under some condition, marginal contributions do not exceed the marginal benefit from the increase of the public project.

Lemma A2 Suppose that $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$ is a Nash equilibrium of Γ^0. Suppose also that $\Delta u_j(y, y-1) \geq 0$ for all $j \in \mathcal{N}$ and all $y \in \mathcal{Y}$ such that $1 \leq y \leq y(\sigma_{\mathcal{N}})$. Then,

$$\Delta u_j(y(\sigma_{\mathcal{N}}), y') \geq \sum_{y=y'+1}^{y(\sigma)} \sigma_j^y \text{ for all } j \in \mathcal{N} \text{ and for all } y' \in \mathcal{Y} \text{ such that } y' \leq y(\sigma_{\mathcal{N}}) - 1. \tag{16}$$

Proof. The proof is obtained by induction. Let $i \in N$. Suppose, to the contrary, that

$$\Delta u_i \left(y(\sigma_{\mathcal{N}}), y(\sigma_{\mathcal{N}}) - 1\right) < \sigma_i^{y(\sigma_{\mathcal{N}})}.$$

Then, $\sigma_i^{y(\sigma_{\mathcal{N}})} > 0$. If i reduces his contribution to the $y(\sigma)$-th unit to zero, the level of the project decreases to $y(\sigma_{\mathcal{N}}) - 1$ by (14) of Lemma A1 and his payoff increases by $\sigma_i^{y(\sigma_{\mathcal{N}})} - \Delta u_i \left(y(\sigma_{\mathcal{N}}), y(\sigma_{\mathcal{N}}) - 1\right) > 0$, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium.

Let $y' \in \mathcal{Y}$ be such that $1 \leq y' \leq y(\sigma_{\mathcal{N}}) - 1$. Suppose, as an induction hypothesis, that $\Delta u_i(y(\sigma_{\mathcal{N}}), y') \geq \sum_{y=y'+1}^{y(\sigma_{\mathcal{N}})} \sigma_i^y$. Then, we show that $\Delta u_i \left(y(\sigma_{\mathcal{N}}), y' - 1\right) \geq \sum_{y=y'}^{y(\sigma_{\mathcal{N}})} \sigma_i^y$. Suppose, to the contrary, that $\Delta u_i(y(\sigma_{\mathcal{N}}), y' - 1) < \sum_{y=y'}^{y(\sigma_{\mathcal{N}})} \sigma_i^y$. By this inequality,

$$\Delta u_i \left(y(\sigma_{\mathcal{N}}), y'\right) - \sum_{y=y'+1}^{y(\sigma_{\mathcal{N}})} \sigma_i^y + \Delta u_i(y', y' - 1) < \sigma_i^{y'}.$$

By this condition and the induction hypothesis, $\Delta u_i(y', y' - 1) < \sigma_i^{y'}$. By $\Delta u_i(y', y' - 1) \geq 0$, we obtain $\sigma_i^{y'} > 0$. Let $\tilde{\sigma}_i \in \mathbb{R}_+^{\bar{y}}$ be i's deviation strategy from σ such that $\tilde{\sigma}_i^y = \sigma_i^y$ if $1 \leq y \leq y' - 1$ and $\tilde{\sigma}_i^y = 0$ otherwise. Since $\sigma_i^{y'} > 0$, the project is undertaken at the level $y' - 1$ at $(\tilde{\sigma}_i, \sigma_{N\setminus\{i\}})$ and $\mathcal{V}_i(\tilde{\sigma}_i, \sigma_{\mathcal{N}\setminus\{i\}}) = u_i(y' - 1) - \sum_{y=1}^{y'-1} \sigma_i^y$. We obtain $\mathcal{V}_i(\sigma_{\mathcal{N}}) - \mathcal{V}_i(\tilde{\sigma}_i, \sigma_{\mathcal{N}\setminus\{i\}}) = \Delta u_i \left(y(\sigma_{\mathcal{N}}), y' - 1\right) - \sum_{y=y'}^{y(\sigma_{\mathcal{N}})} \sigma_i^y < 0$, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium. ■

Lemma A3 provides a sufficient condition of a Nash equilibrium in Γ^0.

Lemma A3 Let $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$. Suppose that for all $j \in \mathcal{N}$,

$$\Delta u_j(y, y - 1) \geq 0 \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y(\sigma_{\mathcal{N}}) \tag{17}$$

$$\text{and } \Delta u_j(y, y(\sigma_{\mathcal{N}})) \leq \Delta c(y, y(\sigma_{\mathcal{N}})) \text{ for all } y \in \mathcal{Y} \text{ such that } y \geq y(\sigma_{\mathcal{N}}) + 1. \tag{18}$$

Suppose also that $\sigma_{\mathcal{N}}$ satisfies (14)–(16). Then, $\sigma_{\mathcal{N}}$ is a Nash equilibrium of Γ^0.

Proof. Let $j \in \mathcal{N}$ and let $\hat{\sigma}_j \in \mathbb{R}_+^{\bar{y}}$ be a deviation strategy of j from σ. First, we consider the case of $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \geq y(\sigma) + 1$. Since by this deviation, the level of the project increases from $y(\sigma)$ to $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$ and $\sigma_{\mathcal{N}}$ satisfies (15), then j contributes at least $\Delta c\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma)\right)$ to this increase. Moreover, by (14), j cannot reduce his contributions from the first to $y(\sigma)$-th unit to undertake the project at the level $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$. Thus,

$$\sum_{y=1}^{\bar{y}} \hat{\sigma}_j^y \geq \sum_{y=1}^{y(\sigma)} \sigma_j^y + \Delta c \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma) \right). \tag{19}$$

By (18), $\Delta c \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma) \right) \geq \Delta u_j \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma) \right)$. Then, by (19),

$$\sum_{y=1}^{\bar{y}} \hat{\sigma}_j^y \geq \sum_{y=1}^{y(\sigma)} \sigma_j^y + \Delta u_j \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma) \right)$$

or, equivalently,

$$u_j(y(\sigma)) - \sum_{y=1}^{y(\sigma)} \sigma_j^y \geq u_j \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \right) - \sum_{y=1}^{\bar{y}} \hat{\sigma}_j^y.$$

In this condition, the left-hand side is the payoff to j before the deviation and the right-hand side is the one after the deviation. Hence, j is not made better off by this deviation.

Second, we consider the case of $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \leq y(\sigma) - 1$. Since $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$ units of the project are undertaken,

$$\sigma_j^y \leq \hat{\sigma}_j^y \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \text{ and } \sigma_j^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})+1} > \hat{\sigma}_j^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})+1}.$$

The maximal payoff to agent j by this deviation is $u_j \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \right) - \sum_{y=1}^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})} \sigma_j^y$, which is obtained if $\hat{\sigma}_j^y = \sigma_j^y$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$ and $\hat{\sigma}_j^y = 0$ for all $y \in \mathcal{Y}$ such that $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) + 1 \leq y \leq \bar{y}$. The payoff to agent j before this deviation is $u_j(y(\sigma_{\mathcal{N}})) - \sum_{y=1}^{y(\sigma_{\mathcal{N}})} \sigma_j^y$, while that after the deviation is at most $u_j \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \right) - \sum_{y=1}^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})} \sigma_j^y$. Clearly,

$$u_j(y(\sigma_{\mathcal{N}})) - \sum_{y=1}^{y(\sigma_{\mathcal{N}})} \sigma_j^y \geq u_j \left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \right) - \sum_{y=1}^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})} \sigma_j^y$$

because $\Delta u_j \left(y(\sigma_{\mathcal{N}}), y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \right) \geq \sum_{y=y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})+1}^{y(\sigma_{\mathcal{N}})} \sigma_j^y$ by (16). Thus, agent j is not made better off by this deviation. ■

A.2 Results of strong Nash equilibria of Γ^0

Similarly to the main text, let $Y(\mathcal{D}) \in \arg\max_{y \in \mathcal{Y}} \sum_{j \in \mathcal{D}} u_j(y) - c(y)$ for all $\mathcal{D} \subseteq \mathcal{N}$ and let $Y^{\max} \equiv \max_{\mathcal{D} \subseteq \mathcal{N}} Y(\mathcal{D})$.

Lemma A4 Let $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$ be a Nash equilibrium such that $y(\sigma_{\mathcal{N}}) = Y^{\max}$. Suppose that

$$\sum_{j \in \mathcal{E}} \Delta u_j(y, Y^{\max}) \leq \Delta c(y, Y^{\max}) \text{ for all } \mathcal{E} \subseteq \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq Y^{\max} + 1. \tag{20}$$

If a coalition $\mathcal{D} \subseteq \mathcal{N}$ has deviation strategies $\sigma'_{\mathcal{D}} \in \mathbb{R}_+^{|\mathcal{D}|\bar{y}}$ such that $y(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}}) \geq Y^{\max} + 1$, then $\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}}) \geq \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$.

Proof. Let $y' \equiv y(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$. Suppose, to the contrary, that $\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}}) > \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}})$. By this inequality,

$$\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) > \sum_{j\in\mathcal{D}} \sum_{y=1}^{Y^{\max}} (\sigma_j'^y - \sigma_j^y) + \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma_j'^y.$$

Since the project is undertaken at the level y' and $Y^{\max} < y'$, we obtain $\sum_{j\in\mathcal{D}} \sum_{y=1}^{Y^{\max}} (\sigma_j'^y - \sigma_j^y) \geq 0$; otherwise, y' units are never provided. Consequently,

$$\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) > \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma_j'^y, \tag{21}$$

On the other hand, by (20), we obtain $\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) \leq \Delta c(y', Y^{\max})$. Since the project is undertaken at the level y' by this deviation and $y' \leq \bar{y}$,

$$\Delta c(y', Y^{\max}) \leq \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{y'} \sigma_i'^y \leq \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma_i'^y.$$

Thus,

$$\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) \leq \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma_i'^y,$$

which contradicts (21). ■

Lemma A5 Suppose that (20) and

$$\Delta u_j(y, y-1) \geq 0 \text{ for all } j \in \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}. \tag{22}$$

Then, every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers of Γ^0.

Proof. Let $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$ be a Nash equilibrium such that $y(\sigma_{\mathcal{N}}) = Y^{\max}$. By (22) and Lemmas A1 and A2,

$$\sum_{j\in\mathcal{N}} \sigma_j^y = \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max},$$

$$\sigma_j^y = 0 \text{ for all } j \in \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq y(Y^{\max}) + 1, \text{ and} \tag{23}$$

$$\Delta u_j(Y^{\max}, \hat{y}) \geq \sum_{y=\hat{y}+1}^{Y^{\max}} \sigma_j^y \text{ for all } j \in \mathcal{N} \text{ and for all } \hat{y} \in \mathcal{Y} \text{ such that } \hat{y} \leq Y^{\max} - 1. \tag{24}$$

Suppose, to the contrary, that there exist a coalition $\mathcal{D} \subsetneq \mathcal{N}$ and $\sigma'_{\mathcal{D}} \in \mathbb{R}_+^{|\mathcal{D}|\bar{y}}$ such that

$$\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}}) < \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}}). \tag{25}$$

Let $y' \in \mathcal{Y}$ be the level of the public project that deviates by $\mathcal{D}$: $y' \equiv y(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$. By Lemma A4, if $y' \geq Y^{\max} + 1$, then it is impossible that $\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}}) < \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$. It is trivial that if $y' = Y^{\max}$, then the deviation by $\mathcal{D}$ is not improving. Finally, we need to consider the case of $y' \leq Y^{\max} - 1$. By (25),

$$\sum_{j\in\mathcal{D}} \Delta u_j(Y^{\max}, y') < \sum_{j\in\mathcal{D}} \sum_{y=1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y).$$

Since the deviation by $\mathcal{D}$ attains y', $\sum_{j\in\mathcal{D}} \sum_{y=1}^{y'} \sigma_j^y \leq \sum_{j\in\mathcal{D}} \sum_{y=1}^{y'} \sigma_j'^y$. By this inequality,

$$\begin{aligned}\sum_{j\in\mathcal{D}} \sum_{y=1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y) &= \sum_{j\in\mathcal{D}} \sum_{y=1}^{y'} (\sigma_j^y - \sigma_j'^y) + \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y) \\ &\leq \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y) \leq \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{\bar{y}} \sigma_j^y = \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y.\end{aligned}$$

The last equality follows from (23). Combining these two conditions yields

$$\sum_{j\in\mathcal{D}} \Delta u_j(Y^{\max}, y') < \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y.$$

However, by (24), $\sum_{j\in\mathcal{D}} \Delta u_j(Y^{\max}, y') \geq \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y$, which is a contradiction.

In conclusion, no coalition can jointly deviate from $\sigma_{\mathcal{N}}$ in a way that improves the sum of payoffs of its members. ■

Appendix B: Proofs of the results in the main text

Proof of Theorem 1. Consider a unit-by-unit contribution game $\Gamma = [N, (S_i, V_i)_{i\in N}]$ in economy $\mathbf{e}^1$. We consider a case in which $\mathcal{N} = N$ and $\mathcal{V}_i = V_i$ for all $i \in N$ and apply Lemmas A1, A3, A4, and A5 to Γ. Claims 1 and 2 are basic properties of economy $\mathbf{e}^1$.

Claim 1 In economy $\mathbf{e}^1$, $Y^{\max} = y^*$.

Proof of Claim 1. By the definition of $Y^{\max}$, $y^* \leq Y^{\max}$. Suppose that $y^* < Y^{\max}$. Since $\{y^*\} = \arg\max_{y\in\mathcal{Y}} \sum_{j\in N} u_j(y) - c(y)$, then $\sum_{j\in N} \Delta u_j(Y^{\max}, y^*) < \Delta c(Y^{\max}, y^*)$. Then, there exists $D \subsetneq N$ such that $Y(D) = Y^{\max}$ and $\sum_{j\in D} \Delta u_j(Y^{\max}, y^*) \geq \Delta c(Y^{\max}, y^*)$. In conclusion,

$$\sum_{j\in D} \Delta u_j(Y^{\max}, y^*) \geq \Delta c(Y^{\max}, y^*) > \sum_{j\in N} \Delta u_j(Y^{\max}, y^*).$$

However, it is impossible that $\sum_{j\in D}\Delta u_j(Y^{\max}, y^*) > \sum_{j\in N}\Delta u_j(Y^{\max}, y^*)$ because $D \subsetneq N$ and $\Delta u_j(y, y-1) \geq 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y \geq 1$. Hence, $Y^{\max} = y^*$ in $\mathbf{e}^1$. ||

Claim 2 In economy $\mathbf{e}^1$, (20) in Lemma A4 holds.

Proof of Claim 2. By Claim 1, $Y^{\max} = y^*$. Since y^* is the unique maximizer of $\sum_{j\in N} u_j(y) - c(y)$, then $\sum_{j\in N}\Delta u_j(y, y^*) < \Delta c(y, y^*)$ for all $y \in \mathcal{Y}$ such that $y \geq y^* + 1$. For all such y, since $\Delta u_j(y, y^*) \geq 0$ for all $j \in N$, we have $\sum_{j\in E}\Delta u_i(y, y^*) \leq \sum_{j\in N}\Delta u_j(y, y^*)$ for all $E \subseteq N$. Thus, for all $E \subseteq N$ and all $y \in \mathcal{Y}$ such that $y \geq y^* + 1$,

$$\sum_{j\in E}\Delta u_i(y, y^*) \leq \sum_{j\in N}\Delta u_j(y, y^*) < \Delta c(y, y^*).$$

Hence, (20) in Lemma A4 holds. ||

Proof of Theorem 1(i). Suppose, to the contrary, that there exists a Nash equilibrium $\sigma \in \mathbb{R}_+^{n\bar{y}}$ such that $y(\sigma) > y^*$. By the efficiency of y^*, $\sum_{j\in N}\Delta u_j(y(\sigma), y^*) < \Delta c(y(\sigma), y^*)$. Since σ is a Nash equilibrium, we obtain $\Delta c(y(\sigma), y^*) = \sum_{j\in N}\sum_{y=y^*+1}^{y(\sigma)}\sigma_j^y$ by (14). By these two conditions, $\sum_{j\in N}\Delta u_j(y(\sigma), y^*) < \sum_{j\in N}\sum_{y=y^*+1}^{y(\sigma)}\sigma_j^y$. This condition implies that there must be $l \in N$ such that $0 \leq \Delta u_l(y(\sigma), y^*) < \sum_{y=y^*+1}^{y(\sigma)}\sigma_l^y$. Now, consider a deviation by agent l such that she makes the same contributions from the first to the y^*-th unit and she makes no contributions to the level over y^*. If we denote the level after such a deviation by y', then $y' \in \{y^*, \ldots, y(\sigma) - 1\}$, and agent l gains $\sum_{y=y^*+1}^{y(\sigma)}\sigma_l^y - \Delta u_l(y(\sigma), y') > 0$,[10] which contradicts the supposition that σ is a Nash equilibrium.

Proof of Theorem 1(ii). First, we show that there is a Nash equilibrium at which the project is undertaken at the level y^*.

Claim 3 There exists $\sigma^* \in \mathbb{R}_+^{n\bar{y}}$ such that

$$\begin{aligned}
&\sigma_j^{*y} = 0 \text{ for all } j \in N \text{ and all } y \in \mathcal{Y} \text{ such that } y > y^*,\\
&\sum_{j\in N}\sigma_j^{*y} = \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y^*, \qquad (26)\\
&\text{and } \Delta u_j(y^*, \hat{y}) \geq \sum_{y=\hat{y}+1}^{y^*}\sigma_j^{*y} \text{ for all } j \in N \text{ and all } y \in \mathcal{Y} \text{ such that } 0 \leq \hat{y} \leq y^* - 1.
\end{aligned}$$

Proof of Claim 3. Obviously, we can set $\sigma_j^{*y} = 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y > y^*$.

We construct $(\sigma_j^{*y})_{j\in N}$ for all y such that $1 \leq y \leq y^*$ by induction. We start with $y = y^*$. By Lemma 1, we obtain $\sum_{j\in N}\Delta u_j(y^*, y^* - 1) > \Delta c(y^*, y^* - 1)$. Thus, there exists $(\sigma_j^{y^*})_{j\in N} \in \mathbb{R}_+^n$ such that

$$\sum_{j\in N}\sigma_j^{*y^*} = \Delta c(y^*, y^* - 1) \text{ and } \Delta u_j(y^*, y^* - 1) \geq \sigma_j^{*y^*} \text{ for all } j \in N.$$

[10] Note that $\Delta u_l(y(\sigma), y') \leq \Delta u_l(y(\sigma), y^*) < \sum_{y=y^*+1}^{y(\sigma)}\sigma_l^y$.

Given this $(\sigma_j^{*y^*})_{j\in N}$, we next construct $(\sigma_j^{*y^*-1})_{j\in N}$.

Let $\hat{y} \in \mathcal{Y}$ be such that $1 \leq \hat{y} \leq y^*-1$. Suppose that $(\sigma_j^{*y})_{j\in N}$ has been constructed for all $y \in \mathcal{Y}$ such that $\hat{y}+1 \leq y \leq y^*$. We now construct $(\sigma_j^{*\hat{y}})_{j\in N}$. By Lemma 1, $\sum_{j\in N} \Delta u_j(y^*, \hat{y}-1) > \Delta c(y^*, \hat{y}-1)$. This condition is equivalent to

$$\sum_{j\in N} \Delta u_j(y^*, \hat{y}) + \sum_{j\in N} \Delta u_j(\hat{y}, \hat{y}-1) > \Delta c(y^*, \hat{y}) + \Delta c(\hat{y}, \hat{y}-1)$$

$$= \sum_{y=\hat{y}+1}^{y^*} \sum_{j\in N} \sigma_j^{*y} + \Delta c(\hat{y}, \hat{y}-1).$$

Thus,

$$\sum_{j\in N} \left[\Delta u_j(y^*, \hat{y}) - \sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \right] + \sum_{j\in N} \Delta u_j(\hat{y}, \hat{y}-1) > \Delta c(\hat{y}, \hat{y}-1)$$

By the induction hypothesis, $\Delta u_j(y^*, \hat{y}) - \sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \geq 0$ for all $j \in N$. Hence, there exists $(\sigma_j^{*\hat{y}})_{j\in N}$ such that

$$\sum_{j\in N} \sigma_j^{*\hat{y}} = \Delta c(\hat{y}, \hat{y}-1) \text{ and } \Delta u_j(y^*, \hat{y}-1) - \sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \geq \sigma_j^{*\hat{y}} \text{ for all } j \in N.$$

||

At σ^*, the project is undertaken at the level y^*. Note that σ^* satisfies all conditions in Lemma A3; hence, it is a Nash equilibrium.

Claim 4 Every Nash equilibrium at which the project is undertaken at y^* is a strong Nash equilibrium with transfers.

Proof of Claim 4. First, note that (22) in Lemma A5 holds since $\Delta u_j(y, y-1) \geq 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y \geq 1$. Second, by Claim 2, (20) in Lemma A4 holds. Thus, by Lemma A5, every Nash equilibrium at which the project is undertaken at the level y^* is a strong Nash equilibrium with transfers. ||

By Claim 4 and Remark 1, with and without transfers, every Nash equilibrium at which the project is undertaken at y^* is a strong Nash equilibrium and a coalition-proof Nash equilibrium. Note that by the definitions of strong Nash equilibria with and without transfers, all strong Nash equilibria with and without transfers must be Nash equilibria at which the project is undertaken at the level y^*. Hence, the sets of these two strong Nash equilibria coincide with the set of Nash equilibrium with efficient implementation of the project.

Claim 5 All coalition-proof Nash equilibria with and without transfers are strong Nash equilibria with transfers.

Proof of Claim 5. First, we show that every coalition-proof Nash equilibrium (without transfers) is a strong Nash equilibrium with transfers. Since the sets of strong Nash equilibria with and without transfers coincide, it is enough to show that without transfers, every coalition-proof Nash equilibrium is a strong Nash equilibrium. Suppose, to the contrary, that there exists a coalition-proof Nash equilibrium $\sigma \in \mathbb{R}_+^{n\bar{y}}$ that is not a strong Nash equilibrium. Since the set of strong Nash equilibria coincides with that of Nash equilibria at which y^* is the level of the project, $y(\sigma)$ must be an inefficient level of the project. Hence, $y(\sigma) < y^*$ and $\sum_{j\in N} \Delta u_j(y^*, y(\sigma)) > \Delta c(y^*, y(\sigma))$. Similarly to the construction of σ^* in (26), we construct $(\sigma_j'^y)_{j\in N}$ for all $y \in \mathcal{Y}$ such that $y(\sigma) + 1 \le y \le \bar{y}$ as follows:

- $\sigma_j'^y = 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y > y^*$.
- $\sum_{j\in N} \sigma_j'^y = \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $y(\sigma) + 1 \le y \le y^*$.
- $\Delta u_j(y^*, \hat{y}) \ge \sum_{y=\hat{y}+1}^{y^*} \sigma_j'^y$ for all $j \in N$ and all $\hat{y} \in \mathcal{Y}$ such that $y(\sigma) \le \hat{y} \le y^* - 1$.
- $\Delta u_j(y^*, y(\sigma)) > \sum_{y=y(\sigma)+1}^{y^*} \sigma_j'^y$ for all $j \in N$.

The last condition follows from $\sum_{j\in N} \Delta u_j(y^*, y(\sigma)) > \Delta c(y^*, y(\sigma))$. Combining $(\sigma_j'^y)_{y=y(\sigma)+1}^{\bar{y}}$ with $(\sigma_j^y)_{y=1}^{y(\sigma)}$ for all $j \in N$, we make a new strategy profile $\tilde{\sigma} \equiv ((\sigma_j^y)_{y=1}^{y(\sigma)}, (\sigma_j'^y)_{y=y(\sigma)+1}^{\bar{y}})_{j\in N}$. At $\tilde{\sigma}$, the project is undertaken at the level y^* and by Lemma A3, it is a Nash equilibrium. By Lemma A5, $\tilde{\sigma}$ is a strong Nash equilibrium with transfers and hence, $\tilde{\sigma}$ is also coalition-proof without transfers. Since $\Delta u_j(y^*, y(\sigma)) > \sum_{y=y(\sigma)+1}^{y^*} \sigma_j'^y$ for all $j \in N$, $\tilde{\sigma}$ Pareto dominates σ. By Definition 2, σ cannot be a coalition-proof Nash equilibrium, which is a contradiction (see Remark 1(iii)).

Finally, note that we can show similarly that every coalition-proof Nash equilibrium with transfers is a strong Nash equilibrium with transfers. ||

In conclusion, we obtain that the five equilibrium sets coincide. ■

Proof of Proposition 1. (⇐) We show $\sigma^* \in \mathbb{R}_+^{n\bar{y}}$ constructed in (8) is a Nash equilibrium. First, note that by (1), (5), and (9),

$$\Delta u_j(y+1, y) \le \Delta c(y+1, y) \text{ for all } j \in N \text{ and all } y \in \mathcal{Y} \text{ such that } y \ge y^*. \tag{27}$$

Considering the game Γ^0 with $\mathcal{N} = N^{y^*}$, we apply Lemma A3 to the game $\Gamma|\sigma^*_{N\setminus N^{y^*}}$. Clearly, we have $\Delta u_j(y, y-1) \ge 0$ for all $j \in N^{y^*}$ and all $y \in \mathcal{Y}$ such that $1 \le y \le y^*$; hence, (17) holds at $\sigma_{\mathcal{N}} = \sigma^*_{N^{y^*}}$. By (27), (18) holds at $\sigma_{\mathcal{N}} = \sigma^*_{N^{y^*}}$. (14)–(16) hold by the construction of $\sigma^*_{N^{y^*}}$. Thus, by Lemma A3, $\sigma^*_{N^{y^*}}$ is a Nash equilibrium of $\Gamma|\sigma^*_{N\setminus N^{y^*}}$.

By (27), no agent outside N^{y^*} is made better off if he unilaterally increases his contribution in such a way that the project is undertaken over the level y^*. Also, no agent outside N^{y^*} is made better off if he increases contributions to the level under y^* because the contributions from agents in N^{y^*} already cover the cost of y^* units. In conclusion, σ^* is a Nash equilibrium of the unit-by-unit contribution game.

(⇒) Suppose, to the contrary, that there exists a Nash equilibrium σ such that $y(\sigma) = y^*$ and that $\Delta u_j(y^*+1, y^*) > \Delta c(y^*+1, y^*)$ for some $j \in N$. By applying Lemma A1, we have $\sigma_i^{y^*+1} = 0$ for all $i \in N$. Clearly, if agent j switches from $\sigma_j^{y^*+1} = 0$ to $\tilde{\sigma}_j^{y^*+1} = \Delta c(y^*+1, y^*)$, then his payoff increases by $\Delta u_j(y^*+1, y^*) - \Delta c(y^*+1, y^*) > 0$, which is a contradiction. ■

Proof of Proposition 2. ($\Rightarrow$) Suppose that there exists a Nash equilibrium σ such that $y(\sigma) > y^*$ but $Y^{\max} \leq y^*$. Since $Y^{\max} < y(\sigma)$, then $Y(D) < y(\sigma)$ for all $D \subseteq N$. Note especially that $Y(N^{y(\sigma)}) < y(\sigma)$.

Since σ is a Nash equilibrium, then $\sigma_j^{y(\sigma)} = 0$ for all $j \in N \backslash N^{y(\sigma)}$: if $\sigma_j^{y(\sigma)} > 0$ for some $j \in N \backslash N^{y(\sigma)}$, agent j is made better off by deviating from σ in a way that changes her contribution to the $y(\sigma)$-th unit to zero and takes the same contribution to the other units as σ_j.[11] Thus,

$$\Delta c(y(\sigma), y(\sigma) - 1) = \sum_{j \in N^{y(\sigma)}} \sigma_j^{y(\sigma)}.$$

By the properties of Nash equilibria in Lemma A2, we obtain $\Delta u_j(y(\sigma), y(\sigma) - 1) \geq \sigma_j^{y(\sigma)}$ for all $j \in N^{y(\sigma)}$, implying

$$\sum_{j \in N^{y(\sigma)}} \Delta u_j(y(\sigma), y(\sigma) - 1) \geq \Delta c(y(\sigma), y(\sigma) - 1).$$

By the weak concavity of u_j and the weak convexity of c, for all $y \in \mathcal{Y}$ such that $y \leq y(\sigma)$,

$$\sum_{j \in N^{y(\sigma)}} \Delta u_j(y, y - 1) \geq \sum_{j \in N^{y(\sigma)}} \Delta u_j(y(\sigma), y(\sigma) - 1) \geq \Delta c(y(\sigma), y(\sigma) - 1) \geq \Delta c(y, y - 1).$$

These inequalities imply that $Y(N^{y(\sigma)}) < y(\sigma)$ never holds; by these inequalities, if $Y(N^{y(\sigma)}) < y(\sigma)$, then $Y(N^{y(\sigma)})$ cannot be a unique maximizer of $\sum_{j \in N^{y(\sigma)}} u_j(y) - c(y)$.

($\Leftarrow$) Let $D \subseteq N$ be such that $Y(D) = Y^{\max}$. By the definition of $Y(D)$, $\sum_{j \in D} \Delta u_j(Y^{\max}, Y^{\max} - 1) > c(Y^{\max}, Y^{\max} - 1)$. Since $\Delta u_j(Y^{\max}, Y^{\max} - 1) < 0$ for all $j \in N \backslash N^{Y^{\max}}$, then $\sum_{j \in D \cap N^{Y^{\max}}} \Delta u_j(Y^{\max}, Y^{\max} - 1) > c(Y^{\max}, Y^{\max} - 1)$. By this inequality, the weak concavity of u_j, and the weak convexity of c,

$$\sum_{j \in D \cap N^{Y^{\max}}} \Delta u_j(y, y - 1) > \Delta c(y, y - 1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}.$$

By this condition, we can construct $\tilde{\sigma} \in \mathbb{R}_+^{n\bar{y}}$ such that

- If $Y^{\max} + 1 \leq y \leq \bar{y}$, then $\tilde{\sigma}_j^y = 0$ for all $j \in N$.
- If $1 \leq y \leq Y^{\max}$, then $\tilde{\sigma}_j^y = 0$ for all $j \in N \backslash (D \cap N^{Y^{\max}})$, $0 \leq \tilde{\sigma}_j^y \leq \Delta u_j(y, y - 1)$ for all $j \in D \cap N^{Y^{\max}}$, and $\sum_{j \in D \cap N^{Y^{\max}}} \tilde{\sigma}_j^y = \Delta c(y, y - 1)$.

Similarly to the method in the proof of Proposition 1, if we apply Lemma A3 to $\Gamma | \tilde{\sigma}_{N \backslash (D \cap N^{Y^{\max}})}$, we can show that $\tilde{\sigma}$ is a Nash equilibrium of Γ.[12] ■

[11] By this deviation, the level of the project decreases to $y(\sigma) - 1$ and j's payoff increases by $-\Delta u_j(y(\sigma), y(\sigma) - 1) + \sigma_j^{y(\sigma)} > 0$.

[12] Note that $\Delta u_j(Y^{\max} + 1, Y^{\max}) \leq \Delta c(Y^{\max} + 1, Y^{\max})$ for all $j \in N$ since $Y^{\max} = \max_{D' \subseteq N} Y(D')$.

Proof of Lemma 5. Let $M \subseteq N$ be such that $Y(M) = Y^{\max}$. Since $\{Y^{\max}\} = \arg\max_{y \in \mathcal{Y}} \sum_{j \in M} u_j(y) - c(y)$,

$$\sum_{j \in M} \Delta u_j(Y^{\max}, Y^{\max} - 1) > \Delta c(Y^{\max}, Y^{\max} - 1). \tag{28}$$

Excluding agents $j \in M$, if any, such that $\Delta u_j(Y^{\max}, Y^{\max} - 1) \leq 0$, we make $M^+ \equiv \{j \in M | \Delta u_i(Y^{\max}, Y^{\max} - 1) > 0\}$. Then, by (28), we obtain

$$\Delta c(Y^{\max}, Y^{\max} - 1) < \sum_{j \in M} \Delta u_j(Y^{\max}, Y^{\max} - 1) \leq \sum_{j \in M^+} \Delta u_j(Y^{\max}, Y^{\max} - 1).$$

By the weak concavity of u_j for all $j \in N$ and the weakly convexity of c,

$$\sum_{j \in M^+} \Delta u_j(y, y - 1) > \Delta c(y, y - 1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}. \tag{29}$$

By the weak concavity of u_j for all $j \in N$, if $\Delta u_j(Y^{\max}, Y^{\max} - 1) > 0$, then $\Delta u_j(y, y - 1) > 0$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y^{\max}$. Hence, M^+ is a set that satisfies (10) and (11). ■

Proof of Proposition 3. We show that σ, constructed in (12), is coalition-proof with and without transfers. Suppose that a coalition $D \subseteq N$ deviates from σ_D to $\sigma'_D \in \mathbb{R}_+^{|D|\bar{y}}$. Let $y' \equiv y(\sigma'_D, \sigma_{N \setminus D})$: y' is the level of the public project attained by this deviation. Trivially, note that the deviation is never profitable if $y' = Y^{\max}$.

Claim 6 Suppose that $y' \geq Y^{\max} + 1$. Then, $\sum_{j \in D} V_j(\sigma) \geq \sum_{j \in D} V_j(\sigma'_D, \sigma_{N \setminus D})$ and there is $i \in D$ such that $V_i(\sigma) \geq V_i(\sigma'_D, \sigma_{N \setminus D})$.

Proof of Claim 6. We prove this claim by Lemma A4. We consider the case of $\Gamma^0 = \Gamma$. Let $E \subseteq N$. Since $Y(E)$ is the unique maximizer of $\sum_{j \in E} u_j(y) - c(y)$,

$$\sum_{j \in E} \Delta u_j(Y(E) + 1, Y(E)) < \Delta c(Y(E) + 1, Y(E)).$$

Thus, by the weak concavity of u_j and the weak convexity of c, for all $y \in \mathcal{Y}$ such that $y \geq Y(E)$,

$$\sum_{j \in E} \Delta u_j(y + 1, y) \leq \sum_{j \in E} \Delta u_j(Y(E) + 1, Y(E)) < c(Y(E) + 1, Y(E)) \leq c(y + 1, y). \tag{30}$$

Note that by the definition of $Y^{\max}$, $Y(E) \leq Y^{\max}$. By this condition and (30),

$$\sum_{j \in E} \Delta u_j(y, Y^{\max}) < c(y, Y^{\max}) \tag{31}$$

for all $y \in \mathcal{Y}$ such that $y \geq Y^{\max} + 1$. Thus, in Γ, (20) holds. By Lemma A4, $\sum_{j \in D} V_j(\sigma) \geq \sum_{j \in D} V_j(\sigma'_D, \sigma_{N \setminus D})$, implying that there exists $i \in D$ such that $V_i(\sigma) \geq V_i(\sigma'_D, \sigma_{N \setminus D})$. ||

First, we show that σ is a coalition-proof Nash equilibrium (without transfers). Suppose, to the contrary, that σ is not coalition-proof. Then, there exist a coalition $D \subseteq N$ and $\sigma'_D \in \mathbb{R}_+^{|D|\bar{y}}$ such that $V_i(\sigma) < V_i(\sigma'_D, \sigma_{N\setminus D})$ for all $i \in D$, and σ'_D is a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N\setminus D}$. By Claim 6, if $y' \geq Y^{\max} + 1$, then it is impossible that the deviation by D is profitable, irrespective of the self-enforcing property of σ'_D. Hence, we need to consider the case of $y' \leq Y^{\max} - 1$.

Claim 7 Suppose that $y' \leq Y^{\max} - 1$. In the restricted game $\Gamma|(\sigma_{N\setminus D}, \sigma'_{D\setminus \mathcal{M}})$,

(7. i) $((\sigma_i'^y)_{y=1}^{y'}, (\sigma_i^y)_{y=y'+1}^{\bar{y}})_{i\in D\cap\mathcal{M}}$, where $(\sigma_i^y)_{y=y'+1}^{\bar{y}}$ is defined in (12) for all $i \in D \cap \mathcal{M}$, is a Nash equilibrium at which the project is undertaken at the level $Y^{\max}$,

(7. ii) every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers, and

(7. iii) $\sigma'_{D\cap\mathcal{M}}$ is strictly Pareto dominated by $((\sigma_i'^y)_{y=1}^{y'}, (\sigma_i^y)_{y=y'+1}^{\bar{y}})_{i\in D\cap\mathcal{M}}$.

Proof of Claim 7. For notational simplicity, denote $\sigma'_{N\setminus(D\cap\mathcal{M})} \equiv (\sigma_{N\setminus D}, \sigma'_{D\setminus\mathcal{M}})$ and $\sigma^{**}_{D\cap\mathcal{M}} \equiv ((\sigma_i'^y)_{y=1}^{y'}, (\sigma_i^y)_{y=y'+1}^{\bar{y}})_{i\in D\cap\mathcal{M}}$.

If the level of the public project declines to y' by this deviation, some agents in $\mathcal{M}$ join in this deviation; otherwise, the level of the project never decreases (note that at σ, no agent outside $\mathcal{M}$ contributes). Hence, $D \cap \mathcal{M} \neq \emptyset$.

Proof of (7.i) We apply Lemma A3 to show that $\sigma^{**}_{D\cap\mathcal{M}}$ is a Nash equilibrium, considering $\Gamma^0 = \Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$; that is, $\mathcal{N}$ in Lemma A3 is equal to $D \cap \mathcal{M}$. Obviously, at $\sigma^{**}_{D\cap\mathcal{M}}$, the project undertaken at $Y^{\max}$ in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$. By (10), (17) holds at $\sigma_{\mathcal{N}} = \sigma^{**}_{D\cap\mathcal{M}}$ in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$. (31) implies that (18) holds at $\sigma_{\mathcal{N}} = \sigma^{**}_{D\cap\mathcal{M}}$ in this game. By the fact that σ'_D must be a Nash equilibrium in $\Gamma|\sigma_{N\setminus D}$, Lemmas A1 and A2, and the construction of σ in (12), $\sigma^{**}_{D\cap\mathcal{M}}$ satisfies (14)–(16) in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$. Hence, by Lemma A3, $\sigma^{**}_{D\cap\mathcal{M}}$ is a Nash equilibrium.

Proof of (7.ii) We show by Lemma A5. We consider Γ^0 in which $\mathcal{N} = \mathcal{D} \cap \mathcal{M}$ and $\mathcal{V}_j(\bullet) = V_j(\bullet, \sigma'_{N\setminus(D\cap\mathcal{M})})$ for all $j \in D \cap \mathcal{M}$. By (10) of Lemma 5, we obtain $\Delta u_j(y, y-1) > 0$ for all $j \in D \cap \mathcal{M}$ and all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y^{\max}$. Hence, (22) holds.

We now prove that (20) holds. Let $E \subseteq D \cap \mathcal{M}$. In a way similar to (30),

$$\sum_{j\in E} \Delta u_j(y+1, y) \leq \sum_{j\in E} \Delta u_j(Y(E)+1, Y(E)) < c(Y(E)+1, Y(E)) \leq c(y+1, y).$$

for all $y \in \mathcal{Y}$ such that $y \geq Y(E)$. In a way similar to (31),

$$\sum_{j\in E} \Delta u_j(y, Y^{\max}) < c(y, Y^{\max}).$$

for all $y \in \mathcal{Y}$ such that $y \geq Y^{\max} + 1$. Thus, (20) holds. By Lemma A5, every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$.

Proof of (7.iii) For all $i \in D \cap \mathcal{M}$, i's payoff at $\sigma'_{D\cap\mathcal{M}}$ is $u_i(y') - \sum_{y=1}^{y'} \sigma_i'^y$, while his payoff at $\sigma^{**}_{D\cap\mathcal{M}}$ is

$$u_i(Y^{\max}) - \sum_{y=1}^{y'} \sigma_i'^y - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y = u_i(y') - \sum_{y=1}^{y'} \sigma_i'^y + \Delta u_i(Y^{\max}, y') - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y.$$

By (13), $\Delta u_i(Y^{\max}, y') - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y > 0$ for all $i \in D \cap \mathcal{M}$; hence,

$$u_i(y') - \sum_{y=1}^{y'} \sigma_i'^y < u_i(Y^{\max}) - \sum_{y=1}^{y'} \sigma_i'^y - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y \tag{32}$$

for all $i \in D \cap \mathcal{M}$. ||

By (7.i) and (7.ii) of Claim 7, $\sigma^{**}_{D\cap\mathcal{M}}$ is a strong Nash equilibrium with transfers of $\Gamma|(\sigma_{N\setminus D}, \sigma'_{D\setminus\mathcal{M}})$; hence, it is a coalition-proof Nash equilibrium of the restricted game. Note that by the definition of coalition-proof Nash equilibria, no coalition-proof Nash equilibrium is Pareto dominated by other coalition-proof Nash equilibria (see Remark 1(iii)). Thus, by (7.iii) of this claim, $\sigma'_{D\cap\mathcal{M}}$ is not coalition-proof in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$, which in turn implies that σ'_D is not a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N\setminus D}$.[13] This is a contradiction. Thus, by Claims 6 and 7, σ is a coalition-proof Nash equilibrium of Γ.

Second, we can similarly show that σ is a coalition-proof Nash equilibrium with transfers. We can prove that if σ'_D is a coalition-proof Nash equilibrium with transfers of $\Gamma|\sigma_{N\setminus D}$, then in $\Gamma|(\sigma_{N\setminus D}, \sigma'_{D\setminus\mathcal{M}})$, $\sigma^{**}_{D\cap\mathcal{M}}$ is a strong Nash equilibrium with transfers at which the project is undertaken at the level $Y^{\max}$ in a similar way to (7.ii) of Claim 7. Moreover,

$$\sum_{j\in D\cap\mathcal{M}} \left(u_j(y') - \sum_{y=1}^{y'} \sigma_j'^y \right) < \sum_{j\in D\cap\mathcal{M}} \left(u_j(Y^{\max}) - \sum_{y=1}^{y'} \sigma_j'^y - \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y \right),$$

which is obtained by summing (32) over $j \in D \cap \mathcal{M}$. By applying a reasoning in Remark 1(iv), σ'_D is not coalition-proof with transfers in $\Gamma|\sigma_{N\setminus D}$. Hence, σ is also coalition-proof with transfers. ■

Proof of Proposition 4. First, note that by Lemma A1, since $\hat{\sigma}$ is a Nash equilibrium, $\sum_{j\in N} \hat{\sigma}_j^y = \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \le y \le y(\hat{\sigma})$ and $\sum_{j\in N} \hat{\sigma}_j^y = 0$ for all $y \in \mathcal{Y}$ such that $y(\hat{\sigma}) + 1 \le y \le Y^{\max}$. Denote $\sigma^{**}_{\mathcal{M}} \equiv ((\hat{\sigma}_i^y)_{y=1}^{y(\hat{\sigma})}, (\sigma_i^y)_{y=y(\hat{\sigma})+1}^{\bar{y}})_{i\in\mathcal{M}}$, in which $((\sigma_i^y)_{y=y(\hat{\sigma})+1}^{\bar{y}})_{i\in\mathcal{M}}$ is defined in (12). As in (7.i) of Claim 7, $\sigma^{**}_{\mathcal{M}}$ is shown to be a Nash equilibrium at which the project is undertaken at $Y^{\max}$ in $\Gamma|\hat{\sigma}_{N\setminus\mathcal{M}}$.

Claim 8 In $\Gamma|\hat{\sigma}_{N\setminus\mathcal{M}}$, (8.i) $V_j(\sigma^{**}_{\mathcal{M}}, \hat{\sigma}_{N\setminus\mathcal{M}}) > V_j(\hat{\sigma})$ for all $j \in \mathcal{M}$ and (8.ii) $\sigma^{**}_{\mathcal{M}}$ is a strong Nash

[13] Note that if σ'_D is a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N\setminus D}$, then σ'_E is also coalition-proof of the corresponding restricted game for all $E \subsetneq D$.

equilibrium with transfers.

Proof of Claim 8. The proof of (8.i) is similar to that of (7.iii) of Claim 7. Note that $\Delta u_j(Y^{\max}, y(\hat{\sigma})) > \sum_{y=y(\hat{\sigma})+1}^{Y^{\max}} \sigma_j^y$ for all $j \in \mathcal{M}$ (see (13)).

(8.ii) is shown by Lemma A5. We consider Γ^0 such that $\mathcal{N} = \mathcal{M}$ and $\mathcal{V}_j(\bullet) = V_j(\bullet, \hat{\sigma}_{N \setminus \mathcal{M}})$ for all $j \in \mathcal{M}$. By (10) of Lemma 5, we have $\Delta u_j(y, y-1) > 0$ for all $j \in \mathcal{M}$ and all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y^{\max}$. Hence, (22) holds.

We now prove that (20) holds. Let $E \subseteq \mathcal{M}$. Since $Y(E)$ uniquely maximizes $\sum_{j \in E} u_j(y) - c(y)$,

$$\sum_{j \in E} \Delta u_j(Y(E)+1, Y(E)) < \Delta c(Y(E)+1, Y(E)).$$

By (1) and (5), for all $y \in \mathcal{Y}$ such that $y \geq Y(E)$,

$$\sum_{j \in E} \Delta u_j(y+1, y) \leq \sum_{j \in E} \Delta u_j(Y(E)+1, Y(E)) < c(Y(E)+1, Y(E)) \leq c(y+1, y).$$

By this condition and $Y(E) \leq Y^{\max}$, we obtain $\sum_{j \in E} \Delta u_j(y, Y^{\max}) < c(y, Y^{\max})$ for all $y \in \mathcal{Y}$ such that $y \geq Y^{\max} + 1$. Thus, (20) holds.

By Lemma A5, every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers. Since $\sigma^{**}_{\mathcal{M}}$ is a Nash equilibrium at which the project is undertaken at $Y^{\max}$, it is also a strong Nash equilibrium with transfers. ||

We can show by Claims 8 that $\hat{\sigma}$ is not coalition-proof with or without transfers in Γ, similarly to the last two paragraphs of the proof of Proposition 3. ■

Proof of Theorem 3. Suppose that $Y^{\max} = y^*$. Then, by Propositions 3 and 4, the level of the project supported at coalition-proof Nash equilibria with and without transfers is y^* or higher. However, by Theorem 2(i), there exists no Nash equilibrium at which the project is undertaken over y^*. Hence, y^* is a unique level of the project supported at coalition-proof Nash equilibria.

Suppose that $Y^{\max} > y^*$. By Proposition 3, there exists a coalition-proof Nash equilibrium at which the project is undertaken over y^*. By Proposition 4, even if there exist Nash equilibria that support a level that is less than or equal to y^*, they are never coalition-proof with or without transfers in Γ. Hence, the public project is excessively undertaken at the coalition-proof Nash equilibria. ■

Proof of Proposition 5. It is enough to provide an example of the economy. We reconsider the economy specified in Example 1. Remember that $\mathcal{Y} = \{0, 1, 2\}$, $c(y) = 10y$ for all $y \in \mathcal{Y}$, $N = \{1, 2\}$, $u_1(1) = 4$, $u_1(2) = 1$, $u_2(1) = 12$, $u_2(2) = 23$, and $y^* = 1$. We show that no $\sigma \in \prod_{j \in N} S_j$ such that $y(\sigma) = 1$ is a Nash equilibrium. Take a strategy profile σ with $y(\sigma) = 1$. Note that $\sum_{j \in N} \sigma_j^2 < 10$, since $y(\sigma) = 1$. We obtain that $V_i(\sigma) = u_i(1) - \sum_{y=1}^{2} t_i^y(\sigma_i^y)$ for each $i \in N$.

First, suppose that $\sigma_i^2 > 0$ for some $i \in N$. Then, the payoff to agent i at σ is $u_i(1) - t_1^1(\sigma_i^1) - \sigma_i^2$. If agent i switches from $\sigma_i^2 > 0$ to $\sigma_i^{2\prime} = 0$ keeping the contribution to the first unit the same, then she can still enjoy the public project at one unit by $\sum_{j\in N} \sigma_j^2 < 10$ and she receives the payoff $u_i(1) - t_1^1(\sigma_i^1)$, which is greater than the payoff before the switch.

Second, suppose that $\sigma_j^2 = 0$ for each $j \in N$. Then, if agent 2 keeps σ_2^1 the same and changes a contribution to the second unit from σ_2^2 to $\tilde{\sigma}_2^2 = \Delta c(2,1)$, then the second unit is provided and he is made better off (note that $\Delta u_2(2,1) > \Delta c(2,1)$).

Finally, suppose that $\sigma_j^2 \leq 0$ for each $j \in N$ and $\sigma_i^2 < 0$ for at least one $i \in N$. Suppose that agent i switches from $\sigma_i^2 < 0$ to $\sigma_i^{2\prime} = 0$ keeping the contribution to the first unit the same. Then, after the switch, the project level is one unit since $\sigma_i^2 + \sigma_k^2 < \sigma_k^2 \leq 0 < 10$, where $k \neq i$. Thus, after the switch, agent 1 can still enjoy one unit of the project and she receives the payoff $u_i(1) - t_1^1(\sigma_i^1)$, which is greater than $u_i(1) - t_1^1(\sigma_i^1) - t_1^2(\sigma_i^2)$.

In conclusion, no $\sigma \in \prod_{j\in N} S_j$ such that $y(\sigma) = 1$ is a Nash equilibrium. ■

Appendix C: Example of coalition-proofness

Consider a three-player game in Table 1.[14] We assume that the payoffs in this table are transferable among members of a coalition if one forms. There are two pure-strategy Nash equilibria: (x_2, y_2, z_1) and (x_1, y_1, z_2). The former is a coalition-proof Nash equilibrium, but not one with transfers. The latter is a coalition-proof Nash equilibrium with transfers, but not coalition-proof Nash equilibrium. Thus, the two sets of coalition-proof Nash equilibria are nonempty and disjoint.

Table 1 Appendix C

	y_1	y_2
x_1	4.5, 2, 0	1, 1, 1
x_2	1, 1, 1	3, 3, 5

z_1

	y_1	y_2
x_1	2, 2, 3	1, 1, 3
x_2	1, 1, 3	0, 0, 3

z_2

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[14] In this matrix, agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each cell is agent 1's payoff, the second is agent 2's, and the third is agent 3's.

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Online Appendix: Core and strong Nash equilibria of the unit-by-unit contribution mechanism in economy $\mathbf{e^1}$

We consider a cooperative game (N, v), where N is the set of players and v is a characteristic function such that $v(D) \equiv \arg\max_{y \in \mathcal{Y}} \sum_{j \in D} u_j(y) - c(y)$ for all nonempty $D \subseteq N$. A payoff profile $(w_j)_{j \in N} \in \mathbb{R}^n_+$ is a core element of this cooperative game if $\sum_{j \in N} w_j = v(N)$ and $\sum_{j \in D} w_j \geq v(D)$ for all nonempty $D \subseteq N$.

Claim A1 Suppose that $\sigma \in \mathbb{R}^{n\bar{y}}_+$ is a Nash equilibrium of Γ in economy $\mathbf{e^1}$ such that $y(\sigma) = y^*$. Then, (i) $\sum_{j \in N} \sigma^y_j = \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y^*$, (ii) $\sigma^y_j = 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y \geq y^*$, and (iii) $\Delta u_j(y^*, y') \geq \sum_{y=y'+1}^{y^*} \sigma^y_j$ for all $y' \in \mathcal{Y}$ such that $y' \leq y^* - 1$.

Proof. Note that $\Delta u_j(y, y-1) > 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y \geq 1$. Considering Γ^0 in which $\mathcal{N} = N$ and $\mathcal{V}_j = V_j$ for all $j \in \mathcal{N}$, we obtain the statements in Claim A1 by Lemmas A1 and A2. ■

Proposition A1 Let $(w_j)_{j \in N} \in \mathbb{R}^n_+$ be a payoff profile such that $w_j = V_j(\sigma)$ for all $j \in N$ for some strong Nash equilibrium $\sigma \in \mathbb{R}^{n\bar{y}}_+$ of the unit-by-unit contribution mechanism in economy $\mathbf{e^2}$. Then, $(w_j)_{j \in N}$ is a core element of (N, v).

Proof. Suppose that $(w_j)_{j \in N} \in \mathbb{R}^n_+$ is a payoff profile at a strong Nash equilibrium, equivalently a Nash equilibrium with the provision of y^* units of the project. Let $\sigma \in \mathbb{R}^{n\bar{y}}_+$ be the Nash equilibrium supporting $(w_j)_{j \in N}$. Since y^* units of the public project are provided at σ, then by Claim A1,

$$\begin{aligned} &\sum_{j \in N} \sum_{y=1}^{y'} \sigma^y_j = c(y') \text{ for all } y' \in \mathcal{Y} \text{ such that } 1 \leq y' \leq y^*, \\ &\sum_{j \in N} \sigma^y_j = 0 \text{ for all } y \in \mathcal{Y} \text{ such that } y \geq y^* + 1, \\ \text{and } &\Delta u_j(y^*, y') - \sum_{y=y'+1}^{y^*} \sigma^y_j \geq 0 \text{ for all } j \in N \text{ and for all } y' \in \mathcal{Y} \text{ such that } y' \leq y^* - 1. \end{aligned} \tag{33}$$

By the third condition of (33), we obtain $w_j = u_j(y^*) - \sum_{y=1}^{y^*} \sigma^y_j \geq 0$ for all $j \in N$ and

$$\sum_{j \in N} w_j = \sum_{j \in N} \left(u_j(y^*) - \sum_{y=1}^{y^*} \sigma^y_j \right) = \sum_{j \in N} u_j(y^*) - c(y^*) = v(N).$$

Let $D \subsetneq N$. Since agents' benefit functions are weakly increasing, we obtain $Y(D) \leq y^*$. By the definition of v, $v(D) = \sum_{j \in D} u_j(Y(D)) - c(Y(D))$. If $Y(D) = y^*$, then $\sum_{j \in D} w_j \geq v(D)$ holds, trivially. If $Y(D) < y^*$, then subtracting $v(D)$ from $\sum_{j\in D} w_j$ yields

$$\begin{aligned}\sum_{j\in D} w_j - v(D) &= \sum_{j\in D} \left(u_j(y^*) - u_j(Y(D))\right) + c(Y(D)) - \sum_{j\in D}\sum_{y=1}^{y^*} \sigma_j^y \\ &= \sum_{j\in D}\left(\Delta u_j\left(y^*, Y(D)\right) - \sum_{y=Y(D)+1}^{y^*} \sigma_j^y\right) + c(Y(D)) - \sum_{j\in D}\sum_{y=1}^{Y(D)} \sigma_j^y.\end{aligned}$$

By the third condition of (33), $\Delta u_j\left(y^*, Y(D)\right) - \sum_{y=Y(D)+1}^{y^*} \sigma_j^y \geq 0$ for all $j \in D$ and by the first condition of (33), $c(Y(D)) - \sum_{j\in D}\sum_{y=1}^{Y(D)} \sigma_j^y \geq 0$. Thus, $\sum_{j\in D} w_j \geq v(D)$. ■