ISSN 0911-1247

Journal of International Economic Studies

(NO.34)
March 2020

Special Issue

THE INSTITUTE OF COMPARATIVE ECONOMIC STUDIES
HOSEI UNIVERSITY

Tokyo, Japan

The Journal of International Economic Studies was first published in 1985 to serve members of Hosei University as a forum on international economic affairs and related fields. However, scholars from outside the University are also urged to submit papers. The Journal concentrates on discussions of international economic issues, particularly those related to the Japanese or other Asian economies.

All those who wish to contribute to this journal should consult the *Instructions for Contributors* printed on the inside of the back cover.

Copies of the journal are distributed free of charge to institutions serving the scholarly community. All communications should be addressed to the editorial secretary, the Institute of Comparative Economic Studies, Hosei University, 4342 Aihara-machi, Machida-shi, Tokyo 194–0298, Japan.

この用紙は再生紙を利用しています。

Journal of International Economic Studies (2020), No.34, 1

Editor's introduction on the special issue "A theory of coalition-proofness and its applications"

Ryusuke Shinohara

I am very glad to introduce the special issue in Journal of International Economic Studies, titled "A theory of coalition-proofness and its applications." The special issue provides the analysis of coalitional behavior in the non-cooperative game theory and its application to public good provision. This issue includes three papers, all of which are the outcomes of the research project "Public economics and political factors: The evaluation of economic policies and the design of economic systems," conducted at the Institute of Comparative Economic Studies, Hosei University from April 2015 to March 2017.

In the second paper "Coalition-proof Nash equilibria and weakly dominated strategies in aggregative games with strategic substitutes: A note," I examine the relation between coalition-proof Nash equilibria (Bernheim et al., 1987) and weakly dominated strategies in games with strategic substitutes (SS) and monotone externalities (ME). I show that in a game with SS and ME, which is introduced by Quartieri and Shinohara (2015) and has a lot of economic games as examples, every coalition-proof Nash equilibrium is a Nash equilibrium in which all players take undominated strategies. I also obtain as a by-product of the main result that the set of Nash equilibria coincides with the set of undominated Nash equilibria in the game. From the results, I conclude that the relation between the coalition-proof Nash equilibrium and weakly dominated strategies in games with SS is completely different from that in games with strategic complements.

In the third paper "Coalitional equilibria in non-cooperative games with strategic substitutes: self-enforcing coalition deviations and irreversibility," I provide a game-theoretic framework which could be applied to many economic phenomena such as cartel formation and public good provision. Introducing a new equilibrium concept of a coalitional equilibrium with restricted deviations, I examine how effectively equilibria based on coalitional stability refine Nash equilibria in games with strategic substitutes and monotone externalities. From the existing equilibria such as coalition-proof Nash equilibria and near-strong Nash equilibria, I can consider several ways of restriction of coalitional deviations. I incorporate two reasonable self-enforcing conditions of coalition deviations, Nash stability and irreversibility, into the coalitional equilibrium and provide a more general analysis than the earlier researches.

In the fourth paper "Undertaking nonharmful or harmful public projects through unit-by-unit contribution: coordination and Pareto efficiency," I examine the implementation of a public project that is nonharmful for all agents as well as a public project that is harmful for some agents through a unit-by-unit contribution mechanism. For a project that is nonharmful for all agents, efficient implementation is supported at one regular Nash equilibrium and several refined Nash equilibria that are stable against coalition deviations. In this sense, this mechanism works well. On the other hand, when the project is harmful for some agents, this mechanism may not have a Nash equilibrium with efficient implementation of the project. Even when such a Nash equilibrium exists, it may not be selected by any of the refined Nash equilibria. Thus, in this case, this mechanism does not work. Our result shows that the merit of the unit-by-unit contribution mechanism reported in the literature is partially extensible to the implementation of a public project.

Finally, I would like to thank everyone who is related to my project. My special thanks are due to Wataru

Kobayashi, Tomomi Miyazaki, Taro Ohno, Kazuki Hiraga, Haruo Kondo, and Yasuhiro Arai, who all are the project members, for constant supports and valuable comments. I hope that the research outcomes build basis for future works.

Journal of International Economic Studies (2020), No.34, 3–10

Coalition-proof Nash equilibria and weakly dominated strategies in aggregative games with strategic substitutes: A note

Ryusuke Shinohara*

Abstract

We examine the relation between coalition-proof Nash equilibrium (Bernheim et al., 1987) and weakly dominated strategies in games with strategic substitutes (SS) and monotone externalities (ME). We show that in σ-interactive games with SS and ME, every coalition-proof Nash equilibrium is a Nash equilibrium with undominated strategies. We also find as a by-product that the set of Nash equilibria coincides with the set of undominated Nash equilibria in those games. The relation between the coalition-proof Nash equilibrium and weakly dominated strategies in games with SS is completely different from that in games with strategic complements.

Keywords: Coalition-proof Nash equilibirum; Undominated strategies; Aggregative games; Strategic substitutes.

JEL classification: **C72; D62**

1 Introduction

A coalition-proof Nash equilibrium, introduced by Bernheim et al. (1987), has been widely applied to many economic games such as oligopoly markets, public good provision, and political competition, voting, and so forth.[1] Hence, clarifying properties of the equilibrium will benefit the economic analysis. In this study, we examine the relation between undominated strategies and coalition-proof Nash equilibria in a game with strategic substitutes. Table 1 provides a simple example in which the coalition-proof Nash equilibrium consists of dominated strategies.

Example 1 Consider the two-player game in Table 1. In the example, (a, c) is coalition-proof, but a (c) is weakly dominated by b (d, respectively).

Shinohara (2019) points out that even under conditions of monotone externalities and strategic complements, which are familiar in the analysis of economics, the coalition-proof Nash equilibrium may be dominated (see Example 1 of Shinohara (2019)). Dekel and Fudenberg (1990) examine the robustness of solutions to payoff perturbations and suggest that refined equilibria should preferably be an undominated Nash equilibrium. From the

*Department of Economics, Hosei University, 4342 Aihara-machi, Machida, Tokyo, 194-0298, Japan. Tel: (81)-42-783-2534. Fax: (81)-42-783-2611. E-mail: ryusukes@hosei.ac.jp

[1]See Thoron (1998), Chowdhury and Sengupta (2004), and Delgado and Moreno (2004) for the application to oligopoly markets, Laussel and Le Breton (1998) and Shinohara (2010a) for the application to public good provision, Messener and Polborn (2007) and Quartieri and Shinohara (2016) for the application to voting and political competition.

Table 1: A coalition-proof Nash equilibrium consists of weakly dominated strategies

1 \ 2	c	d
a	3, 3	1, 3
b	3, 1	2, 2

viewpoint of Dekel and Fudenberg (1990), Shinohara (2019) provides a new equilibrium concept called the *undominated coalition-proof Nash equilibrium*, which incorporates the undominated-strategy property in coalition-proofness. He shows the existence and the uniqueness of the equilibrium in a game with conditions of strategic *complements* and monotone externalities.

The focus of this study is on the relation between undominated strategies and coalition-proof Nash equilibria in a game with strategic *substitutes*. We consider this relation on the class of σ-interactive games with strategic substitutes and monotone externalities. This class of games is introduced by Quartieri and Shinohara (2015) and it includes games that are frequently studied in economic analysis (see Quartieri and Shinohara, 2015). Quartieri and Shinohara (2015) show the equivalence between the coalition-proof Nash equilibrium and the Nash equilibrium in this class of games. However, whether the coalition-proof Nash equilibrium consists of undominated strategies has not been studied.

We show that in every σ-interactive game with strategic substitutes and monotone externalities, every coalition-proof Nash equilibrium consists of undomiated strategies. This is shown in an interesting way mediated with the undominated coalition-proof Nash equilibrium of Shinohara (2019). First, we show that in the game, the set of Nash equilibria and that of undominated coalition-proof Nash equilibria coincide (Lemma 1). Second, by using the first result, we show that the set of coalition-proof Nash equilibria coincides with the set of undominated Nash equilibria (Proposition 1). As a by-product of the first and second results, we find that the sets of the Nash equilibrium, the undominated Nash equilibrium, the coalition-proof Nash equilibrium, and the undominated coalition-proof Nash equilibrium all coincide in this game (Corollary 1).

The relation between the coalition-proof Nash equilibrium and undominated strategies have been studies by several researchers. Moreno and Wooders (1996) and Milgrom and Roberts (1996) investigate the relation of the equilibrium with the iterative elimination of *strictly* dominated strategies and show that if there exists a profile of serially undominated strategies that Pareto-dominates the other serially undominated strategies, it is a coalition-proof Nash equilibrium. In contrast, the working paper by Shinohara (2010b) examines the relation between the coalition-proof Nash equilibrium and the iterative elimination of *weakly* dominated strategies. Shinohara (2010b) clarifies that when the iterative elimination of weakly dominated strategies is adopted, a Pareto-superior serially undominated Nash equilibrium is not necessarily coalition-proof. His contribution is to establish a sufficient condition under which the coalition-proof Nash equilibrium survives the iterative elimination of weakly dominated strategies. By applying his result, we find that if a game satisfies strategic substitutes and monotone externalities and further the set of strategies is finite for every player, then the coalition-proof Nash equilibrium consists of undominated strategies. The result of the present study generalizes his result because the games considered in the present study satisfies more general conditions of strategic substitutes and monotone externalities and, more importantly, we do not assume that the strategy set is finite. Peleg (1998) examines the relation between the equilibrium and dominant strategies. Pointing out that the coalition-proof Nash equilibrium may consist of weakly dominated strategies, he shows that almost all dominant-strategy equilibria are coalition-proof; thus, such equilibria consist of undominated strategies. In our class of games, a dominant-strategy equilibrium does not necessarily exist.

The remainder of this paper is organized as follows. Section 2 presents the preliminaries. Section 3 provides the results and Section 4 concludes the paper.

2 The model

2.1 Strategic substitutes and monotone externalities in σ-interactive games

A strategic-form game is a list $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, in which N is a finite and nonempty set of players and, for each $i \in N$, $S_i \neq \emptyset$ is the set of strategies of player i and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is player i's payoff function.[2] A subset of N is called a *coalition*. For each coalition $C \subseteq N$, the set of strategy profiles for coalition C is denoted by $S_C \equiv \prod_{i\in C} S_i$. A typical element of S_C is denoted by s_C. Using this notation, we can express $s = \left(s_C, s_{N\setminus C}\right)$ for each $s \in S_N$. If a coalition is a singleton (that is, $C = \{i\}$ for some $i \in N$), then we simply denote its strategy profile by s_i and its set of strategy profiles S_i. Hereafter, the complement of coalition $\{i\}$ is denoted by $-i$, not $N\setminus\{i\}$, for simplicity.

Let $b_i : S_N \to 2^{S_i}$ denote the *best response correspondence* of player $i \in N$: For each $s \in S_N$,

$$b_i(s) \equiv \arg\max_{z\in S_i} u_i(z, s_{-i}).$$

We do not restrict the player's best response strategies to being unique.

We focus on a σ-*interactive* game, which is defined as follows:

Definition 1 A game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a σ-interactive game if

1. $S_i \subseteq \mathbb{R}$ for each $i \in N$ and
2. For each $i \in N$, there exists a function $\sigma_i : S_N \to \mathbb{R}$ such that σ_i is non-decreasing in s_j $(j \neq i)$ and constant in s_i; for all $s, \tilde{s} \in S_N$, if $s_i = \tilde{s}_i$ and $\sigma_i(s) = \sigma_i(\tilde{s})$, then $u_i(s) = u_i(\tilde{s})$.

In this game, players' strategies are real numbers. For each player $i \in N$, σ_i "aggregates" strategies of the players other than i. The aggregated value of the strategies through σ_i, not their composition, affects player i's payoff. One of the examples of function σ_i is $\sigma_i(s) = \sum_{j\neq i} s_j$. Under this function, player i's payoff depends on its own strategy s_i and the sum of the others' strategies, as in the standard Cournot oligopoly game and the public good game.

We consider a σ-interactive game in which every player's best response correspondence is "non-increasing" and players' payoff functions are monotonic.

Definition 2 A σ-interactive game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ satisfies σ-*interactive strategic substitutes* (σ-SS) if for all $i \in N$, all $s, t \in S_N$, all $v_i \in b_i(s)$, and all $w_i \in b_i(t)$,

$$\text{if } \sigma_i(s) < \sigma_i(t), \text{ then } v_i \geq w_i.$$

Definition 3

- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies σ-*increasing externalities* (σ-IE) if for all $s, t \in S_N$ and all $i \in N$, if $s_i = t_i$ and $\sigma_i(s) \leq \sigma_i(t)$, then $u_i(s) \leq u_i(t)$. A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies σ-*decreasing externalities* (σ-DE) if $[N, S_N, (-u_i)_{i\in N}]$ is a game with σ-IE.
- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies σ-*monotone externalities* (σ-ME) if Γ satisfies σ-IE or σ-DE.

The conditions of strategic substitutes and monotone externalities are defined in terms of the aggregated value by σ_i. σ-SS requires that the best response strategies for player $i \in N$ are non-increasing with regard to aggregated values by function σ_i. σ-ME requires that the payoff function of player $i \in N$ is monotonic with regard to aggregated values by function σ_i.

[2]The model in this study is based on that in Quartieri and Shinohara (2015).

Our focus is limited to pure-strategies. Quartieri and Shinohara (2015) present several examples of games of economic interest that satisfy σ-SS and σ-ME. They show examples that have multiple pure-strategy Nash equilibria. Hence, the class of our games also possibly includes games with multiple equilibria.

2.2 Equilibrium concepts

The Nash equilibrium is defined as usual.

Definition 4 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. A strategy profile $s \in S_N$ is a *Nash equilibrium* (NE) for Γ if $s_i \in b_i(s)$ for all $i \in N$. The set of Nash equilibria for Γ is denoted by $NE(\Gamma)$.

In Definition 5, we introduce the notions of undominated strategies and Nash equilibria.

Definition 5

- In $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, $s_i \in S_i$ is *weakly dominated* by $t_i \in S_i$ if $u_i(t_i, z_{-i}) \geq u_i(s_i, z_{-i})$ for all $z_{-i} \in S_{-i}$ and $u_i(t_i, z_{-i}) > u_i(s_i, z_{-i})$ for some $z_{-i} \in S_{-i}$. Strategy s_i is *undominated* in Γ if no player i's strategy weakly dominates s_i. Let $\tilde{S}_i$ be the set of player i's undominated strategies in Γ.
- A strategy profile $s \in S_N$ is a *undominated Nash equilibrium* (UNE) for Γ if s_i is undominated for all $i \in N$ and s is a Nash equilibrium for Γ. The set of undominated Nash equilibria for Γ is denoted by $UNE(\Gamma)$.

For preparation to introduce coalition-proof Nash equilibria, we introduce a notion of induced games.

Definition 6 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$. For all $C \subsetneq N$ and all $s \in S_N$, the game $\Gamma|s_{-C} = (C, (S_i)_{i\in C}, (\tilde{u}_i)_{i\in C})$ is the *game induced by C at s* in which $\tilde{u}_i : S_C \to \mathbb{R}$ is the payoff function of player $i \in C$ such that $\tilde{u}_i(t_C) \equiv u_i(t_C, s_{-C})$ for all $t_C \in S_C$.

A *coalition-proof Nash equilibrium*, introduced in Bernheim et al. (1987), is as follows. This is recursively defined with regard to the number of players in coalitions by using the induced games.

Definition 7 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. If $|N| = 1$, then $s \in S_N$ is a coalition-proof Nash equilibrium (CP-NE) for Γ if and only if $s \in NE(\Gamma)$. As the induction hypothesis, we assume that $|N| \geq 2$ and that a CP-NE has been defined for games with fewer than $|N|$ players. Then,

- $s \in S_N$ is a *self-enforcing strategy* for Γ if it is a CP-NE for $\Gamma|s_{-C}$ for all nonempty $C \subsetneq N$;
- $s \in S_N$ is a CP-NE for Γ if it is self-enforcing for Γ and there does not exist another self-enforcing strategy $t \in S_N$ for Γ that strongly Pareto dominates s in Γ: $u_i(s) < u_i(t)$ for all $i \in N$.

The set of coalition-proof Nash equilibria in Γ is denoted by $CPNE(\Gamma)$.

We now introduce an *undominated coalition-proof NE* (UCP-NE) by incorporating the condition that players take undominated strategies into the original definition of CP-NE (Bernheim et al., 1987). The equilibrium is introduced by Shinohara (2019).

Definition 8 An undominated coalition-proof Nash equilibrium (UCP-NE) for Γ is defined by induction with respect to the number of members in a coalition. First, define a UCP-NE for single-player coalitions.

1. Let $i \in N$ and $s_{-i} \in S_{-i}$. Strategy $s_i^* \in S_i$ is a UCP-NE for $\Gamma|s_{-i}$ if $s_i^* \in \arg\max_{t_i \in S_i} u_i(t_i, s_{-i})$ and $s_i^* \in \tilde{S}_i$.

Next, define a UCP-NE for a coalition with more than one player.

2. Let C be such that $|C| \geq 2$, and let $s_{-C} \in S_{-C}$. As an induction hypothesis, a UCP-NE is defined in the restricted games in which D is the set of players for all $D \subsetneq C$.

   (a) $s^*_C \in S_C$ is *undominated self-enforcing* (U-self-enforcing) *for* $\Gamma|s_{-C}$ if, for all $D \subsetneq C$, s^*_D is a UCP-NE for $\Gamma|(s^*_{C\setminus D}, s_{-C})$ and $s^*_i \in \tilde{S}_i$ for all $i \in C$.

   (b) s^*_C is a UCP-NE for $\Gamma|s_{-C}$ if s^*_C is U-self- enforcing in $\Gamma|s_{-C}$ and there is no U-self-enforcing $t_S \in S_C$ for $\Gamma|s_{-C}$ such that $u_i(t_C, s_{-C}) > u_i(s^*_C, s_{-C})$ for all $i \in C$.

If $C = N$, $s^* \in S_N$ is defined as a UCP-NE for Γ. Let $UCPNE(\Gamma)$ be the set of undominated coalition-proof Nash equilibria for Γ.

Remark 1 We immediately obtain the following properties from the definitions of the equilibria.

(i) In every game Γ, every coalition-proof Nash equilibrium is a Nash equilibrium: $CPNE(\Gamma) \subseteq NE(\Gamma)$.

(ii) In every game Γ, every undominated Nash equilibrium is a Nash equilibrium: $UNE(\Gamma) \subseteq NE(\Gamma)$.

(iii) In every game Γ, every undominated coalition-proof Nash equilibrium is a Nash equilibrium: $UCPNE(\Gamma) \subseteq UNE(\Gamma) \subseteq NE(\Gamma)$.

(iv) Let $s \in S_N$ be a strategy profile and $C \subsetneq N$ be a coalition. Let $t_C \in S_C$ be a UCP-NE in $\Gamma|s_{-C}$. Then, t_C must be a Nash equilibrium in $\Gamma|s_{-C}$.

(v) $CPNE(\Gamma)$ and $UCPNE(\Gamma)$ are not always related by inclusion relation (see, for example, Example 1 of Shinohara (2019)).

3 Results

Lemma 1 shows the equivalence between Nash equilibria and uncominated coalition-proof Nash equilibria for any σ-interactive game with σ-SS and σ-ME.

Lemma 1 If $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a σ-interactive game with σ-SS and σ-ME, then

$$NE(\Gamma) = UCPNE(\Gamma).$$

Proof. Suppose that IE holds. From the definitions of the Nash equilibrium and UCP-NE, it is immediately apparent that every UCP-NE is a Nash equilibrium in Γ. Next, we show that every Nash equilibrium is a UCP-NE. Suppose, to the contrary, that $s \in S_N$ is a Nash equilibrium, but not a UCP-NE for Γ. Then, there exist a coalition $C \subseteq N$ and a U-self-enforcing strategy profile $t_C \in S_C$ such that

$$u_i(t_C, s_{-C}) > u_i(s) \text{ for all } i \in C. \tag{1}$$

Suppose that there exists $i \in C$ such that $\sigma_i(t_C, s_{-C}) \leq \sigma_i(s)$. Then, $\sigma_i(s) = \sigma_i(t_i, s_{-i})$ since σ_i is constant in i's strategies. We find that $u_i(s) \geq u_i(t_i, s_{-i})$ since s is a Nash equilibrium. We further find from $\sigma_i(t_i, s_{-i}) \geq \sigma_i(t_C, s_{-C})$ and IE that $u_i(t_i, s_{-i}) \geq u_i(t_C, s_{-C})$. Finally, we obtain the result that $u_i(s) \geq u_i(t_C, s_{-C})$, which contradicts (1). Thus, it follows that

$$\sigma_i(t_C, s_{-C}) > \sigma_i(s) \text{ for all } i \in C. \tag{2}$$

Suppose that $t_i \leq s_i$ for all $i \in C$. Then, by the non-decreasing property of σ_i, we find that $\sigma_i(t_C, s_{-C}) \leq \sigma_i(s)$ for all $i \in C$, which contradicts (2). Thus,

$$\text{there exists } j \in C \text{ such that } t_j > s_j. \tag{3}$$

By the definition of UCP-NE, t_C must be a Nash equilibrium for $\Gamma|s_{-C}$ (see (iii) and (iv) of Remark 1). Thus, for the player j, t_j is a best reply to (t_C, s_{-C}): $t_j \in b_j(t_C, s_{-C})$. Finally, from σ-SS, we obtain the result that $s_j \in b_j(s)$, $t_j \in b_j(t_C, s_{-C})$, and $\sigma_j(t_C, s_{-C}) > \sigma_j(s)$ imply $s_j \geq t_j$. This contradicts (3).

The proof when DE holds is similar. ■

Proposition 1 If $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a σ-interactive game with σ-SS and σ-ME, then

$$CPNE(\Gamma) = UNE(\Gamma).$$

Proof. By Theorem 1 of Quartieri and Shinohara (2015), $NE(\Gamma) = CPNE(\Gamma)$ holds. In addition, by Lemma 1, $NE(\Gamma) = CPNE(\Gamma) = UCPNE(\Gamma)$ holds. Finally, by the defitions of UCP-NE, NE and UNE, $UCPNE(\Gamma) \subseteq UNE(\Gamma) \subseteq NE(G) = UCPNE(\Gamma)$ (see Remark 1). Thus, $UNE(\Gamma) = CPNE(\Gamma)$. ■

We finally obtain the following corollary immediately from Lemma 1, Proposition 1, and the result of Quartieri and Shinohara (2015).

Corollary 1 If $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a σ-interactive game with σ-SS and σ-ME, then

$$NE(\Gamma) = CPNE(\Gamma) = UCPNE(\Gamma) = UNE(\Gamma).$$

σ-SS and -ME are crucial for the property that every coalition-proof Nash equilibrium always consists of undominated strategies, which is exemplified in the following examples.

Example 1 continued. In this example, we additionally assume that $a, b, c, d \in \mathbb{R}$ such that $a > b$ and $c > d$. Then, σ-IE is satisfied while σ-SS is not satisfied. As we already see, strategy profile (a, c) is a unique CP-NE, but a and c are weakly dominated.

Example 2 Consider the game in Table 2, in which $a, b, c, d, e, f \in \mathbb{R}$ and $a < b < c$ and $d < e < f$. This game satisfies σ-SS, but does not satisfy σ-ME. Strategy profile (a, f) is the only CP-NE, but it consists of weakly dominated strategies. Strategies b and e are also weakly dominated strategies.

Table 2: Example 2

1 \ 2	d	e	f
a	0, 40	40, 40	40, 40
b	10, 41	45, 40	40, 35
c	20, 38	50, 30	40, 20

Finally, we discuss the difference in the results between games with strategic complements and games with strategic substitutes. Whether CP-NE consists of undominated strategies in games with strategic complements is examined by Shinohara (2019). He examines this in the framework of quasi-super modular games, which are games with strategic complements because the best response correspondence of every player is non-decreasing with regard to the other players' strategies in the games. We summarize the results of Shinohara (2019) as follows:

(C.1) Weakly dominated strategies may constitute a CP-NE.

(C.2) The set of CP-NE and that of UCP-NE both exist, but they may not be related by inclusion.

(C.3) The set of UCP-NE is a subset of the set of UNE. However, they do not necessarily coincide.

We can observe the above three points from Example 1 of Shinohara (2019).

In contrast to the above properties, we obtain the following properties in games with strategic substitutes:

(S.1) Every CP-NE always consists of undominated strategies.

(S.2) The set of CP-NE and that of UCP-NE always coincide.

(S.3) The set of UCP-NE always coincides with the set of UNE.

Thus, the relation between CP-NE and undominated strategies is completely different between games with strategic complements and games with strategic substitutes. If we consider that the refinement of Nash equilibria should consist of undominated strategies, as Dekel and Fudenberg (1990) discuss, then CP-NE satisfies this property in games with strategic substitutes. Unlike in the games with strategic complements, we do not have to consider UCP-NE.

4 Conclusion

We examine whether coalition-proof Nash equilibria take undominated strategies in games with strategic substitutes and monotone externalities. In contrast to the results in games with strategic complements by Shinohara (2019), we show that every coalition-proof Nash equilibrium is an undominated Nash equilibrium; hence, the coalition-proof Nash equilibrium never consists of weakly dominated strategies in games with strategic substitutes. We also find as a by-product that the set of Nash equilibria coincides with that of undominated Nash equilibria in those games.

Although the conditions of strategic substitutes and strategic complements capture many situations which are frequently examined in the economic analysis, there are also many games of economic applications which cannot be captured by those two conditions. Thus, as a future work, it would be interesting to explore the relation between the coalition-proof Nash equilibrium and the undominated strategies in other classes of games.

Acknowledgments I gratefully acknowledge the financial support from the KAKENHI Grant-in-Aid for Scientific Research (C) No. 18K01519.

References

[1] Bernheim, D., Peleg, B., Whinston, M. (1987) Coalition-proof Nash equilibria I: Concepts. Journal of Economic Theory, vol. 42, 1-12.

[2] Chowdhury, P. R. and Sengupta, K. (2004) Coalition-proof Bertrand equilibria. Economic Theory, vol. 24, 307-324.

[3] Dekel, E., Fudenberg, D. (1990) Rational behavior with payoff uncertainty. Journal of Economic Theory, vol. 52, 243–267.

[4] Delgado, J. and Moreno, D. (2004) Coalition-proof supply function equilibria in oligopoly. Journal of Economic Theory, vol. 114, 231-254.

[5] Laussel, D. and Le Breton, M. (1998) Efficient private production of public goods under common agency. Games and Economic Behavior, vol. 25, 194-218.

[6] Messner, M. and Polborn, M.K. (2007) Strong and coalition-proof political equilibria under plurality and runoff rule. International Journal of Game Theory, vol. 35, 287-314.

[7] Moreno, D. and Wooders, J. (1996) Coalition-proof equilibrium. Games and Economic Behavior, vol. 17, 80-112.

[8] Milgrom, P., Roberts, J. (1996) Coalition-proofness and correlation with arbitrary communication possibilities. Games and Economic Behavior, vol. 17, 113-128.

[9] Peleg, B. (1998) Almost all equilibria in dominant strategies are coalition-proof. Economics Letters, vol. 60, 157-162.

[10] Quartieri, F., Shinohara, R. (2015) Coalition-proofness in a class of games with strategic substitutes. International Journal of Game Theory, vol. 44, 785-813.

[11] Quartieri, F. and Shinohara, R. (2016) Two-group contests with communication within and between groups. In: von Mouche P. and Quartieri F. (eds) Equilibrium Theory for Cournot Oligopolies and Related Games, Springer Series in Game Theory (Official Series of the Game Theory Society), Springer, Cham, 245-268.

[12] Shinohara, R. (2010a) Coalition-proof equilibria in a voluntary participation game. International Journal of Game Theory, vol. 12, 603-615.

[13] Shinohara, R. (2010b) Coalition-proof Nash equilibrium of aggregative games. Available at http://ryusukeshinohara.ehoh.net/wp/agg.pdf.

[14] Shinohara, R. (2019) Undominated coalition-proof Nash equilibria in quasi-supermodular games with monotonic externalities. Economics Letters, vol. 176, 86-89.

[15] Thoron, S. (1998) Formation of a coalition-proof stable cartel. Canadian Journal of Economics, vol. 31, 63-76.

Journal of International Economic Studies (2020), No.34, 11–22

Coalitional equilibria in non-cooperative games with strategic substitutes: Self-enforcing coalition deviations and irreversibility*

Ryusuke Shinohara†

Abstract

Introducing a *coalitional equilibrium with restricted deviations*, we examine how effectively equilibria based on coalitional stability refine Nash equilibria in games with σ-strategic substitutes and σ-monotone externalities. From the existing equilibria such as coalition-proof Nash equilibria and near-strong Nash equilibria, we can consider several ways to restrict coalitional deviations. We incorporate two natural self-enforcing conditions of coalition deviations, *Nash stability* and *irreversibility*, into the coalitional equilibrium and provide a more general analysis than earlier studies. We find it impossible that in each of the two stability concepts, the coalitional equilibrium effectively refines the Nash equilibrium for all games with σ-strategic substitutes and σ-monotone externalities.

Keywords: Coalitional equilibrium with restricted deviations; Nash stability; Irreversibility.

JEL classification: **C72; D62**

1 Introduction

We study the refinement of Nash equilibria in a strategic-form game with strategic substitutes (SS) and monotone externalities (ME). Since this game has many examples of economic games such as the Cournot oligopoly game and the game of the private provision of public goods, it is important from the viewpoint of applied game theory to clarify characteristics of the equilibria of this game. The Nash equilibrium, the standard equilibrium concept of the strategic-form game, is not necessarily uniquely determined in this game.[1] Hence, we apply “coalitional refinements” of the Nash equilibrium to the game.

Yi (1999) is the first study to apply the *coalition-proof Nash equilibrium* (Bernheim et al., 1987) to a class of games with SS and ME. Yi (1999) shows that every Pareto-undominated pure-strategy Nash equilibrium is coalition-proof. Shinohara (2010) shows that in the same game, the set of coalition-proof Nash equilibria coincides with the entire set of pure-strategy Nash equilibria. Quartieri and Shinohara (2015) clarify many properties of the coalition-proof Nash equilibria in σ-interactive games with σ-strategic substitutes (σ-SS) and σ-monotone externalities (σ-ME), which generalize Yi’s (1999) and Shinohara’s (2010) games. Quartieri and Shinohara (2015) show that the set of coalition-proof Nash equilibria under strong Pareto dominance (sCPN equilibria, for short) and the entire set of Nash equilibria coincide in these games. They also examine coalition-proof Nash equilibria under weak Pareto dominance (wCPN equilibria, for short) and show that the set of wCPN equilibria also coincides with the set of Nash

*This article is a translated version of R. Shinohara (2019) “Senryakutekidaitai gemu niokeru teikeikinkou: Teikeiridatsu no jikokousokusei to fukagyakusei,” in R. Shinohara eds. *Koukyoukeizaigaku to seijitekiyouin–Keizaiseisaku seido no hyouka to sekkei*, Nippon Hyoron Sha, 37-56 (printed in Japanese). I would like to express deep appreciation to the Institute of Comparative Economic Studies, Hosei University (the copyright holder of the Japanese version of this article) for the permission to translate the article.

†Department of Economics, Hosei University, 4342 Aihara-machi, Machida, Tokyo, 194-0298, Japan. Tel: (81)-42-783-2534. Fax: (81)-42-783-2611. E-mail: ryusukes@hosei.ac.jp

[1] See Quartieri and Shinohara (2015) for examples of games that have multiple pure-strategy Nash equilibria.

equilibria if the best reply correspondence of all players is at most singleton-valued in the same games. Another familiar equilibrium to refine the Nash equilibria is a *strong Nash equilibrium* (Aumann, 1959). However, since the strong Nash equilibrium is too demanding, the set of strong Nash equilibria may be empty, although the set of Nash equilibria is nonempty in the games of Quartieri and Shinohara (2015). Therefore, it seems difficult that the familiar equilibria based on coalitional stability single out a particular Nash equilibrium from multiple Nash equilibria for games with σ-SS and σ-ME.

In this study, we examine whether the equilibrium based on coalitional stability that is both weaker than the strong Nash equilibrium and stronger than the coalition-proof Nash equilibrium effectively refines the Nash equilibrium. Some "intermediate" equilibrium concepts already exist. We can take a *semi-strong Nash equilibrium* (Kaplan, 1992; Milgrom and Roberts, 1994) and a *near-strong Nash equilibrium* (Rozenfeld and Tennenholtz, 2010) as examples of such equilibria.

What is new in this study is the introduction of a new concept of *coalitional equilibria with restricted deviations*, which makes it possible to unify the analysis with the intermediate equilibria. The coalitional equilibrium with restricted deviations is a non-cooperative equilibrium that is stable only against some restricted deviations. The restricted deviations consist of the set of feasible coalitions and feasible deviation strategies for each feasible coalition. They capture the idea that for geographical, legal, or political reasons and so on, not every player can communicate with each other and coalitions that can form are restricted; each feasible coalition faces a self-enforcing problem and its feasible deviation strategies are surely restricted in order for it to execute the deviation. The merit of the coalitional equilibrium with restricted deviations is that we can adequately restate several familiar equilibrium concepts by setting the structure of feasible coalitions and that of feasible deviation strategies, which is formally stated in Proposition 1 below.

We impose an admissible condition on the structure of feasible coalitions, so that deviations by each individual player is possible. We impose two natural self-enforceabilities for deviation strategies, *Nash stability* (NS) and *irreversibility* (IR). NS requires that the deviation strategies of each feasible coalition must be a Nash equilibrium in the game induced by taking players' strategies outside the coalition as fixed. IR requires that the deviation strategies of each feasible coalition must be robust to switching-back options: after the deviation, no member of the coalition switches back to the strategy before deviation, taking the others' strategies as fixed.[2]

We examine how effectively the coalitional equilibria under NS and IR refine Nash equilibria. We first show that under the admissible structure of feasible coalitions, the set of the coalitional equilibria with NS coincides with the set of Nash equilibria in every game with σ-SS and σ-ME. Hence, the coalitional equilibria with NS does not refine the Nash equilibria. We second show that in games with σ-single crossing property, which is stronger than σ-SS, and σ-ME, the set of the coalitional equilibria with IR coincides with the set of Nash equilibria. While there is an example of a game with σ-SS and σ-ME in which the coalitional equilibrium with IR refines the Nash equilibrium, as Example 1 shows, there is a set of games with the same conditions in which Nash equilibria are multiple and the coalitional equilibria with IR does not refine the Nash equilibrium.

We conclude that under NS, which seems to be acceptable as coalitional self-enforceability in non-cooperative games, it is impossible that the coalitional equilibrium provides effective refinements of the Nash equilibrium for games with σ-SS and σ-ME that have multiple Nash equilibria. If we would like to single out some particular Nash equilibria among all the equilibria, we must make the self-enforcing requirement weaker than the NS. The IR is one of the examples. However, it is another problem whether or not we accept the IR or weaker concepts, which do not satisfy the NS, as a self-enforcing requirement because the NS can be considered as a "minimal requirement" for self-enforceability of coalition deviations. Therefore, to refine the Nash-equilibrium analysis through the coalitional equilibria, we must apply self-enforcing conditions, which are mathematically definable but may be unjustifiable as "natural" coalitional behavior in economic meaning.

[2]As we will see later, the set of self-enforcing deviations in w and sCPN equilibria satisfy NS and IR. The set of feasible deviations in near-strong Nash equilibria satisfy IR.

Related literature
Ichiishi (1981) introduces a *social coalitional equilibrium*, which includes the Nash equilibrium and the core of cooperative games with nonsidepayments as special cases. In Ichiishi's (1981) equilibrium, each feasible coalition faces a restriction of deviation strategies. The feasible deviation strategies of a coalition depends on strategies of players outside the coalition, as ours does. However, Ichiishi (1981) does not consider suitable notions of coalitional self-enforceability, unlike ours. Also, in his equilibrium, the coalition that can be formed is *not* restricted: deviations by any coalition are possible. Zhao (1992) introduces the hybrid solution, which can also express the Nash equilibrium and the core by setting coalition structures appropriately. Laraki (2009) introduces an equilibrium concept called a *coalitional equilibrium*. Like ours, in his equilibrium, the coalition formation is restricted. However, unlike ours, the deviation strategies of each coalition are not restricted. His is equivalent with ours if each feasible coalition can take all joint strategies in our equilibrium. In this sense, ours is more general than Laraki's. Finally, we would like to add that Ichiishi (1981), Zhao (1992), and Laraki (2009) focus on the existence of equilibria, but not on their characterization of it. The coalitional refinements of Nash equilibria have been well studied for games with strategic *complements*. See Milgrom and Roberts (1996), Quartieri (2013), and Shinohara (2019).

2 Preliminaries

2.1 Strategic substitutes and monotonic externalities in σ-interactive games

A strategic-form game is a list $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, in which N is a finite and nonempty set of players and, for each $i \in N$, $S_i \neq \emptyset$ is player i's strategy set and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is player i's payoff function.[3] A subset of N is called a *coalition*. For each coalition $C \subseteq N$ and each strategy profile $s \in \prod_{i\in N} S_i$, the set of strategy profiles for coalition C, $\prod_{i\in C} S_i$, is denoted by S_C. A typical element of S_C is denoted by s_C. Using this notation, we can express $s = (s_C, s_{N\backslash C})$ for each $s \in S_N$. If a coalition is a singleton (that is, $C = \{i\}$ for some $i \in N$), then we simply denote its strategy profile by s_i and its set of strategy profiles S_i. Hereafter, the complement of coalition $\{i\}$ is denoted by $-i$, not $N\backslash\{i\}$, for simplicity.

For the game Γ, the *best response correspondence* of player $i \in N$ is defined as $b_i : S_N \to 2^{S_i}$ such that

$$b_i : s \mapsto \arg\max_{z\in S_i} u_i (z, s_{-i}) .$$

The game on which we focus satisfies *σ-interactivity*, which is defined as follows:

Definition 1 A game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a σ-interactive game if and only if

1. $S_i \subseteq \mathbb{R}$ for each $i \in N$ and
2. For each $i \in N$, there exists a function $\sigma_i : S_N \to \mathbb{R}$ such that σ_i is non-decreasing in s_j $(j \neq i)$ and constant in s_i; for all $s, \tilde{s} \in S_N$, if $s_i = \tilde{s}_i$ and $\sigma_i(s) = \sigma_i(\tilde{s})$, then $u_i(s) = u_i(\tilde{s})$.

σ-strategic substitutes and σ-single-crossing property exhibit non-increasing properties of the best response function with regard to strategies of the rival players.

Definition 2 A σ-interactive game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ satisfies *σ-interactive strategic substitutes* (σ-SS) if and only if for all $(x, y, i) \in S_N \times S_N \times N$,

$$z_i \in b_i (x), w_i \in b_i (y) \text{ and } \sigma_i (x) < \sigma_i (y) \text{ implies } w_i \leq z_i.$$

Definition 3 A game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-single crossing property* (σ-SCP) if and only if for all $x, y \in S_N$ and all $i \in N$, if $x_i < y_i$, $\sigma_i(x) < \sigma_i(y)$, and $u_i(x) - u_i(y_i, x_{-i}) \geq 0$, then $u_i(x_i, y_{-i}) - u_i(y) > 0$.

[3]The model in this study is based on that in Quartieri and Shinohara (2015).

Note that for each game, σ-SCP implies σ-SS, but the converse is not true. A game in Example 1 below satisfies σ-SS, but not σ-SCP.

Definition 4

- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-increasing externalities* (σ-IE) if and only if for all $x, y \in S_N$ and all $i \in N$, if $x_i = y_i$ and $\sigma_i(x) \leq \sigma_i(y)$, then $u_i(x) \leq u_i(y)$.
- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-decreasing externalities* (σ-DE) if and only if $[N, S_N, (-u_i)_{i\in N}]$ is a game with σ-IE.
- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-monotone externalities* (σ-ME) if and only if Γ satisfies σ-IE or σ-DE.

Our focus is limited to pure-strategies. Quartieri and Shinohara (2015) present several examples of games of economic interest that satisfy σ-SS and σ-ME. They show examples that have multiple pure-strategy Nash equilibria. Hence, the class of our games also possibly includes games with multiple equilibria.

2.2 Preliminary results on coalition-proofness

The Nash equilibrium is defined as usual.

Definition 5 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. A strategy profile $s \in S_N$ is a *Nash equilibrium* for Γ if and only if $s_i \in b_i(s)$ for all $i \in N$. The set of Nash equilibria in Γ is denoted by E_N^Γ.

Pareto domination among strategy profiles are also usual.

Definition 6 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. A strategy profile $s \in S_N$ *strongly Pareto dominates* in Γ a strategy profile $z \in S_N$ if and only if $u_i(z) < u_i(s)$ for all $i \in N$. The *s-efficient subset of* E_N^Γ is the set of Nash equilibria for Γ that are not strongly Pareto dominated in Γ by other Nash equilibria for Γ. The s-efficient subset of E_N^Γ is denoted by sF_N^Γ.

For preparation, we introduce a notion of induced games.

Definition 7 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. Let $C \in 2^N \setminus \{\emptyset\}$, $s \in S_N$, and for all $i \in C$, $\tilde{u}_i : S_C \to \mathbb{R}$, $\tilde{u}_i : z \mapsto u_i(z, s_{-C})$. The *game induced by C at s* is the game $(C, (S_i)_{i\in C}, (\tilde{u}_i)_{i\in C})$ and is denoted by $\Gamma|s_{-C}$.

A *coalition-proof Nash equilibrium*, introduced in Bernheim et al. (1987), is as follows.

Definition 8 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. If $|N| = 1$, then $s \in S_N$ is an s-coalition-proof Nash equilibrium for Γ if and only if $s \in E_N^\Gamma$. Now assume that $|N| \geq 2$ and that an s-coalition-proof Nash equilibrium has been defined for games with fewer than $|N|$ players. Then,

- $s \in S_N$ is an *s-self-enforcing strategy* for Γ if and only if it is an s-coalition-proof Nash equilibrium for $\Gamma|s_{-C}$ for all nonempty $C \subset N$;
- $s \in S_N$ is an *s-coalition-proof Nash equilibrium* for Γ if and only if it is s-self-enforcing for Γ and there does not exist another s-self-enforcing strategy for Γ that strongly Pareto dominates s in Γ.

The set of s-coalition-proof Nash equilibria in Γ is denoted by E_{sCPN}^Γ.

By definition, it is clear that $E_{sCPN}^\Gamma \subseteq E_N^\Gamma$ and $sF_N^\Gamma \subseteq E_N^\Gamma$ for all Γ. As pointed out by Bernheim et al. (1987), sF_N^Γ and E_{sCPN}^Γ are not related by inclusion for some Γ. However, Quartieri and Shinohara (2015) show the equivalence between these three sets in games with σ-SS and σ-ME.

Result 1 (Quartieri and Shinohara, 2015) *Let Γ be a σ-interactive game with σ-strategic substitutes and monotone externalities. Then,*

(1.1) $E^{\Gamma}_{N} = E^{\Gamma}_{sCPN} = sF^{\Gamma}_{N}$.

(1.2) *If b_i is single-valued for each $i \in N$, then the set of Nash equilibria and the set of coalition-proof Nash equilibria with weak domination coincide.*[4]

The results suggest that in games with strategic substitutes and monotone externalities, it seems very problematical that the coalition-proof Nash equilibrium refines the set of Nash equilibria when they are multiple. Hence, our question moves to whether some other equilibrium concepts, which are stronger than the coalition-proof Nash equilibrium, refine the Nash equilibrium.

2.3 Coalitional equilibria with restricted deviations

We introduce a new concept, called a *coalitional equilibrium with restricted deviations*. We provide a general notion of restriction of coalition deviations such that the coalition deviations can be restricted to express some earlier equilibrium concepts.

For a game $\Gamma = \big(N, (S_j)_{j\in N}, (u_j)_{j\in N}\big)$, $\mathcal{C} \subseteq 2^N\backslash\{\emptyset\}$ is defined as a nonempty set of feasible coalitions: only the coalitions in $\mathcal{C}$ can deviate. For each $D \in \mathcal{C}$ and each $s \in S_N$, denote the set of strategies that coalition D can take when deviating from s by R^s_D. Denote $\mathcal{R}_D \equiv (R^s_D)_{s\in S_N}$ for each $D \in \mathcal{C}$ and $\mathcal{R}_{\mathcal{C}} \equiv (\mathcal{R}_D)_{D\in\mathcal{C}}$. We term a pair $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ the set of *feasible deviations*.

Definition 9 Let $\Gamma = \big(N, (S_j)_{j\in N}, (u_j)_{j\in N}\big)$ be a game. Let $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ be the set of feasible deviations. $s \in S_N$ is a $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$*-coalitional equilibrium* in Γ if there do not exist $D \in \mathcal{C}$ and $\tilde{s}_D \in R^s_D$ such that $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ for each $i \in D$. The set of $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$-coalitional equilibrium in Γ is denoted by $E^{\Gamma}_{(\mathcal{C},\mathcal{R}_{\mathcal{C}})}$.

Next, we introduce a few conditions for the set of feasible deviations. First of all, we introduce the notion of admissibility, which requires that every player can deviate by using every strategy available to it. This requirement seems very natural since each player is assumed to freely choose its strategies in noncooporative games.

Definition 10 $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ is *admissible* if for each $i \in N$ and each $s \in S_N$, $\{i\} \in \mathcal{C}$ and $R^s_i = S_i$.

Henceforth, we assume that admissibility is satisfied.

The *Nash stability* for coalition deviations, defined as follows, seems reasonable under admissibility, because agreed coalition deviations must be immune to the deviation by single members of the coalition under the situation in which every player can take every strategy by admissibility.

Definition 11 Let $\Gamma = \big(N, (S_j)_{j\in N}, (u_j)_{j\in N}\big)$ be a game. $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ satisfies *Nash stability* (NS) if for each $D \in \mathcal{C}$ and each $s \in S_N$,

$$R^s_D \subseteq E^{\Gamma|s_{-D}}_{N} = \{s'_D \in S_D |\, u_i(s'_D, s_{-D}) \geq u_i(s''_i, s'_{D\backslash\{i\}}, s_{-D}) \text{ for each } i \in D \text{ and each } s''_i \in S_i\}.$$

A stability notion weaker than the NS is also introduced as follows:

Definition 12 Let $\Gamma = \big(N, (S_j)_{j\in N}, (u_j)_{j\in N}\big)$ be a game. $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ satisfies *irreversibility* (IR) if for each $D \in \mathcal{C}$ and each $s \in S_N$,

$$R^s_D \subseteq \{s'_D \in S_D |\, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\backslash\{i\}}, s_{-D}) \text{ for each } i \in D\}.$$

[4]The coalition-proof Nash equilibria with weak domination can be defined as in Definition 8 by replacing strong Pareto dominance with weak Pareto dominance. See, for instance, Shinohara (2005) and Quartieri and Shinohara (2015) for the precise definition.

Denote $(C, \mathcal{R}_C)$ satisfying NS and that satisfying IR by $(C, \mathcal{R}_C^{NS})$ and $(C, \mathcal{R}_C^{IR})$, respectively. These two notions of stability capture the idea of "self-enforceability." That is, although each feasible coalition can freely deviate from certain strategies, no member of feasible coalitions can enforce the other members to take and stick to certain deviation strategies. Hence, for a coalition deviation to be done assuredly, the deviation must be "stable" against any further deviation by proper coalition. NS assumes that once a coalition deviates, then each individual member of the coalition deviates further if he/she has a strategy that improves his/her payoff after the original deviation. Under NS, each feasible coalition can conduct the deviations immune to such further deviations. IR is based on the idea that if a coalition deviates, then each individual member of the coalition has an option to withdraw from the deviation and switch back to the original strategy. Under IR, each feasible coalition deviates in such a way that no member executes such an option. Clearly, if $\mathcal{R}_C$ satisfies NS, then it also satisfies IR. Hence, for each C and for each Γ, if a strategy profile is a $(C, \mathcal{R}_C^{IR})$-coalitional equilibrium in Γ, then it is a $(C, \mathcal{R}_C^{NS})$-coalitional equilibrium in Γ; however, the converse is not true.

Proposition 1 summarizes the relation between the $(C, \mathcal{R}_C)$-coalitional equilibrium and several well-known non-cooperative equilibria.

Proposition 1 Let Γ be a game and let $(C, \mathcal{R}_C)$ be an admissible set of feasible deviations.

(1) $E^{\Gamma}_{(C,\mathcal{R}_C)} \subseteq E^{\Gamma}_N$. Further, if $C = \{\{j\} | j \in N\}$, then $E^{\Gamma}_{(C,\mathcal{R}_C)} = E^{\Gamma}_N$.

(2) If $C = 2^N \setminus \{\emptyset\}$ and $R^s_D = S_D$ for each $D \in C$ and each $s \in S_N$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is equivalent with the *strong Nash equilibrium* (Aumann, 1959).[5]

(3) If $C = \{N, \{j\}_{j \in N}\}$ and $R^s_D = E^{\Gamma|s_{-D}}_N$ for each $D \in C$ and each $s \in S_N$, then $E^{\Gamma}_{(C,\mathcal{R}_C)} = sF^{\Gamma}_N$.

(4) If $C = 2^N \setminus \{\emptyset\}$ and $R^s_D = E^{\Gamma|s_{-D}}_N$ for each $D \in C$ for each $s \in S_N$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is equivalent with the *semi-strong Nash equilibrium* in Γ (Kaplan, 1992; Milgrom and Roberts, 1994).[6] Under the same C and R^s_D for each $D \in C$ and each $s \in S_N$, $(C, \mathcal{R}_C)$-coalitional equilibrium is an s-coalition-proof Nash equilibrium in Γ.

(5) If $C = 2^N \setminus \{\emptyset\}$ and $R^s_D = \{s'_D \in S_D | \, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\setminus\{i\}}, s_{-D})$ for each $i \in D\}$ for each $s \in S_N$ and each $D \in C$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is equivalent with the *near-strong Nash equilibrium* (Rozenfeld and Tenneholz, 2010).[7]

Proof of Proposition 1 is in the Appendix.

Remark 1 Milgrom and Roberts (1996) incorporate another notion of restricted coalition formation into the coalition-proof Nash equilibrium. They define a coalition deviation process as a finite sequence of coalitions $\sigma = (C_1, \ldots, C_m)$ such that m is a positive integer and $C_m \subsetneq \cdots \subsetneq C_1 \subseteq N$: this sequence indicates that C_1 can communicate to deviate from a strategy profile; once C_1 has deviated, then C_2 can plan a further deviation from the C_1's deviation, and so on. The set of such sequences, generically denoted by Σ, is called a *coalition communication structure* (CCS). Milgrom and Roberts (1996) impose CCS on some admissibility conditions, which implies that every coalition in the sequences take a Nash equilibrium in the corresponding induced game, and they define a *coalition-proof Nash equilibrium with CCS* along the sequences in Σ, recursively. As in Definition 8, each feasible coalition designated by CCS takes a self-enforcing deviation when deviating. By the definition of the coalition-proof Nash equilibria with CCS, the self-enforcing deviations must be a Nash equilibrium in the corresponding induced game. Thus,

[5] A strategy profile $s \in S_N$ is a strong Nash equilibrium if and only if there is no pair $(D, \tilde{s}_D) \in 2^N \setminus \{\emptyset\} \times S_D$ such that $u_i(s) < u_i(\tilde{s}_D, s_{-D})$ for each $i \in D$.

[6] A strategy profile $s \in S_N$ is a semi-strong Nash equilibrium if and only if there is no pair $(D, \tilde{s}_D) \in 2^N \setminus \{\emptyset\} \times E^{\Gamma|s_{-D}}_N$ such that $u_i(s) < u_i(\tilde{s}_D, s_{-D})$ for each $i \in D$.

[7] A strategy profile $s \in S_N$ is a *near-strong Nash equilibrium* if there is no pair $(D, \tilde{s}_D) \in 2^N \setminus \{\emptyset\} \times S_D$ such that for each $i \in D$, $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ and $u_i(\tilde{s}_D, s_{-D}) \geq u_i(\tilde{s}_{D\setminus\{i\}}, s_{-D\cup\{i\}})$.

if $\mathcal{C} = \{D \subseteq N | D$ is the first element of some $\sigma \in \Sigma\}$ and $R^s_D = E^{\Gamma|s_{-D}}_N$ for each $s \in S_N$ and each $D \in \mathcal{C}$, then $(\mathcal{C}, \mathcal{R}_\mathcal{C})$-coalitional equilibrium is a coalition-proof Nash equilibrium with Σ.[8]

3 Results

The following lemma is a result commonly used in the proof of Propositions 2 and 3.

Lemma 1 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a σ-interactive game with σ-ME. For each $s \in E^\Gamma_N$ and each $\tilde{s} \in S_N$, if $u_i(\tilde{s}) > u_i(s)$ for each $i \in N$, then $\sigma_i(s) < \sigma_i(\tilde{s})$ for each $i \in N$ when Γ is a game with σ-IE and $\sigma_i(s) > \sigma_i(\tilde{s})$ for each $i \in N$ when Γ is a game with σ-DE.

Proof. We provide a proof in the case of σ-IE. The case of σ-DE is similar. Suppose that there exists $j \in N$ such that $u_j(\tilde{s}) > u_j(s)$ and $\sigma_j(s) \geq \sigma_j(\tilde{s})$. Since $s \in E^\Gamma_N$, then $u_j(s) \geq u_j(\tilde{s}_j, s_{-j})$. Since σ_j is constant in the j-th argument, then $\sigma_j(s) = \sigma_j(\tilde{s}_j, s_{-j}) \geq \sigma_j(\tilde{s})$. By σ-IE, $u_j(\tilde{s}_j, s_{-j}) \geq u_j(\tilde{s})$. Thus, $u_j(\tilde{s}) \leq u_j(s)$, which is a contradiction. ■

Note that this lemma is irrelevant to σ-SS.

3.1 Coalitional equilibria with NS

Proposition 2 Suppose that Γ is a σ-interactive game with σ-SS and σ-ME, $\mathcal{C}$ is an admissible set of feasible coalitions, and $\mathcal{R}_\mathcal{C}$ are feasible deviations satisfying NS. Then, $E^\Gamma_{(\mathcal{C},\mathcal{R}_\mathcal{C})} = E^\Gamma_N$.

Proof. Consider games with σ-IE. The proof for the games with σ-DE is similar. By part (1) of Proposition 1, $E^\Gamma_{(\mathcal{C},\mathcal{R}_\mathcal{C})} \subseteq E^\Gamma_N$. We show the converse. Suppose, to the contrary, that there exists $s \in E^\Gamma_N \backslash E^\Gamma_{(\mathcal{C},\mathcal{R}_\mathcal{C})}$. Then, $D \in \mathcal{C}$ and $\tilde{s}_D \in E^{\Gamma|s_{-D}}_N$ exist such that $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ for each $i \in D$. Since $s \in E^\Gamma_N$ and $\tilde{s}_D \in E^{\Gamma|s_{-D}}_N$, then $s_i \in b_i(s)$ and $\tilde{s}_i \in b_i(\tilde{s}_D, s_{-D})$ for each $i \in D$. Applying Lemma 1 to $\Gamma|s_{-D}$ yields $\sigma_i(\tilde{s}_D, s_{-D}) > \sigma_i(s)$ for each $i \in D$. Since σ_i is non-decreasing in all arguments, then the last inequality implies that there exists $i^* \in D$ such that $\tilde{s}_{i^*} > s_{i^*}$. However, by σ-SS, $\sigma_{i^*}(\tilde{s}_D, s_{-D}) > \sigma_{i^*}(s)$, $s_{i^*} \in b_{i^*}(s)$, and $\tilde{s}_{i^*} \in b_{i^*}(\tilde{s}_D, s_{-D})$ imply $\tilde{s}_{i^*} \leq s_{i^*}$, which is a contradiction. ■

By Proposition 2, in games with σ-SS and σ-ME, the $(\mathcal{C}, \mathcal{R}^{NS}_\mathcal{C})$-coalitional equilibrium exists whenever the Nash equilibrium does. However, no $(\mathcal{C}, \mathcal{R}^{NS}_\mathcal{C})$-coalitional equilibrium refines the set of Nash equilibria. As we see in (4) of Proposition 1, the semi-strong Nash equilibrium, which is stronger than the s-coalition-proof Nash equilibria, can be expressed by a $(\mathcal{C}, \mathcal{R}^{NS}_\mathcal{C})$-coalitional equilibrium with some $(\mathcal{C}, \mathcal{R}^{NS}_\mathcal{C})$. Even if we use an equilibrium concept that is stronger than the s-coalition-proof Nash equilibrium, if it is based on the NS, then it never refines the Nash equilibrium in games with σ-SS and σ-ME. This result points out the difficulty of refining the Nash equilibria by the equilibrium based on NS.

This result stems from the order structure of the set of Nash equilibria in games with σ-SS. As Quartieri and Shinohara (2015) show in their Theorem 2, in each game Γ with σ-SS, it is impossible that $\sigma_i(s) < \sigma_i(\tilde{s})$ for all $i \in N$ and all distinct $s, \tilde{s} \in E^\Gamma_N$. However, by Lemma 1, in each game with σ-IE (resp. σ-DE), s strongly Pareto dominates $\tilde{s}$ only if $\sigma_i(s) < \sigma_i(\tilde{s})$ (resp. $\sigma_i(s) > \sigma_i(\tilde{s})$) for each $i \in N$. These apply to any game induced by any $s' \in S$ and any $D \subseteq N$. Therefore, NS and coalitional profitability are incompatible in games with σ-SS and σ-ME.

3.2 Coalitional equilibria with IR

We examine whether a coalitional equilibrium with IR, which is stronger than that with NS, refines the Nash equilibrium in σ-interactive games with σ-SS and σ-ME. Proposition 3 shows that given $\mathcal{C}$, the coalitional equilibria with IR do not refine the Nash equilibria in a proper subclass of σ-interactive games with σ-SS and σ-ME.

[8] Shinohara (2010) shows that the set of coalition-proof Nash equilibria with CCS coincides with the entire set of Nash equilibria in games with strategic substitutes and monotone externalities. See Proposition 2 of Shinohara (2010).

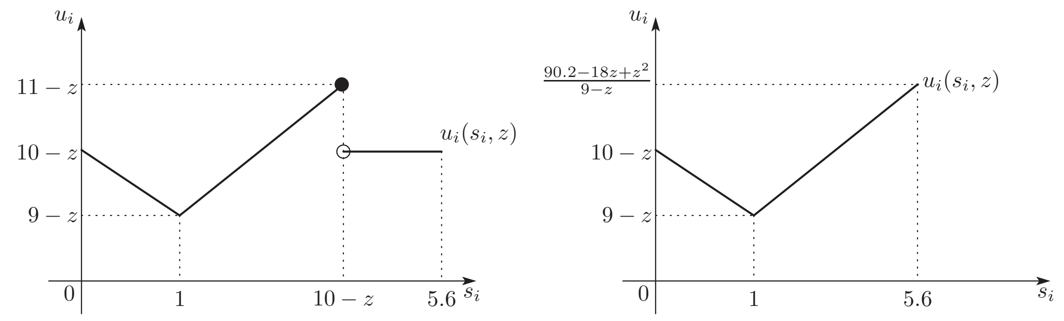

Figure 1: The graph of u_i. The left figure is the case of $z > 4.4$ and the right figure is the case of $z \leq 4.4$.

Proposition 3 Suppose that Γ is a σ-interactive game with σ-SCP and σ-ME, C is an admissible set of feasible coalitions, and $\mathcal{R}_C$ represents feasible deviations satisfying IR. Then, $E^{\Gamma}_{(C,\mathcal{R}_C)} = E^{\Gamma}_N$.

Proof. We treat the case of σ-IE. The case of σ-DE is similar. By part (1) of Proposition 1, $E^{\Gamma}_{(C,\mathcal{R}_C)} \subseteq E^{\Gamma}_N$. We show the converse. Suppose, to the contrary, that there exists $s \in E^{\Gamma}_N \backslash E^{\Gamma}_{(C,\mathcal{R}_C)}$. Then, there exists $D \in C$ and $\tilde{s}_D \in R^s_D$ such that for each $i \in D$, (a) $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ and (b) $u_i(\tilde{s}_D, s_{-D}) \geq u_i(s_i, \tilde{s}_{D\backslash\{i\}}, s_{-D})$. By (a), applying Lemma 1 to $\Gamma|s_{-D}$ yields $\sigma_i(s) < \sigma_i(\tilde{s}_D, s_{-D})$ for each $i \in D$. By this condition, we have $s_{i^*} < \tilde{s}_{i^*}$ for some $i^* \in D$. Since s is a Nash equilibrium, then $u_{i^*}(s) - u_{i^*}(\tilde{s}_i, s_{-i}) \geq 0$. By the σ-SCP, $s_{i^*} < \tilde{s}_{i^*}$, and $\sigma_{i^*}(s) < \sigma_{i^*}(\tilde{s}_D, s_{-D})$, we reveal that $u_{i^*}(s_i, \tilde{s}_{D\backslash\{i\}}, s_{-D}) - u_{i^*}(\tilde{s}_D, s_{-D}) > 0$, which is a contradiction with (b). ■

The set of σ-interactive games with σ-SCP is a proper subset of the set of games with σ-SS. For example, see Example 1, which provides a game satisfying σ-SS but not σ-SCP. The implication of the result is that in games with σ-SCP and σ-ME, the coalitional equilibrium with IR exists whenever a Nash equilibrium exists. However, the coalitional equilibrium with IR never refines the Nash equilibrium.

However, the following example shows the possibility that the coalitional equilibrium with IR works as a refinement of the Nash equilibrium in games with σ-SS and σ-ME but not σ-SCP.

Example 1 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be such that $N = \{1, 2\}$ and for each $i \in N$, $S_i = [0, 5.6]$ and

$$u_i(s_i, s_j) = \begin{cases} 10 - s_i - s_j & \text{if } s_i \in [0, 1] \\ \dfrac{2}{9 - s_j} s_i + \dfrac{79 - 18s_j + s_j^2}{9 - s_j} & \text{if } s_i \in [1, \min\{10 - s_j, 5.6\}] \\ 10 - s_j & \text{if } s_i \in \big(\min\{10 - s_j, 5.6\}, 5.6\big] \text{ and } s_j > 4.4 \end{cases}, \tag{1}$$

where $i \neq j$. Suppose that $C = 2^N \backslash \{\emptyset\}$ and $R^s_D = \{s'_D \in S_D | \, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\backslash\{i\}}, s_{-D})$ for each $i \in D\}$ for each $D \in C$ and each $s \in S_N$. We denote a typical graph of u_i when fixing $s_j = z$ in Figure 1.

Fact 1 Let $\sigma_i(s) = s_j$ for each pair $i, j \in N$ such that $i \neq j$ and each $s \in S_N$. This game is then a game with σ-SS and σ-DE, but not σ-SCP.

Fact 2 It follows that $\emptyset \neq E^{\Gamma}_{(C,\mathcal{R}_C)} \subsetneq E^{\Gamma}_N$.

Proofs of Facts 1 and 2 are in the Appendix.

By Proposition 2, no coalitional equilibria based on Nash stable coalitional deviations refine the Nash equilibria: hence, the s-coalition-proof Nash equilibria and the semi-strong Nash equilibria are not refinements of Nash equilibria. In addition, since the best response correspondence of each player is singleton-valued, then the set of coalition-proof Nash equilibria under *weak Pareto domination* does not refine the Nash equilibria either (see Result

1-(1.2)).[9] Of course, no strong Nash equilibrium exists. Thus, by this example, we can point out that in games with σ-SS and σ-ME, but not σ-SCP, a coalitional equilibrium with IR may provide a refinement of the Nash equilibrium, although the equilibrium concepts which are frequently used in economics can not refine the Nash equilibrium.[10]

4 Conclusion

Introducing the coalitional equilibrium with restricted deviations, we examine how effectively equilibria based on coalitional stability refine Nash equilibria in σ-interactive games with strategic substitutes and monotone externalities. The coalitional equilibrium with restricted deviations can express several familiar equilibria as special cases by setting feasible coalition deviations appropriately. Thereby, we can provide a unified analysis for the issue.

We impose two stability conditions (NS and IR) on feasible coalition deviations. First, we have shown that the set of the coalitional equilibria with NS coincides with the set of Nash equilibria in every σ-interactive game with σ-SS and σ-ME. Hence, the coalitional equilibria with NS does not refine the Nash equilibria in that game. Second, we have pointed out the possibility that the coalitional equilibrium with IR, which is stronger than the equilibrium with NS, singles out a particular Nash equilibrium from all Nash equilibria in that game. We observe this possibility in σ-interactive games that satisfy σ-SS, but not σ-SCP (see Example 1).

If no member of a coalition can force other members to take certain deviation strategies, then whether the coalition deviation is possible depends on whether it is “self-enforcing”. As discussed previously, requiring the NS on coalition deviations seems reasonable in non-cooperative games because the NS is immune to all single-member deviations of the coalition. Hence, we can consider the NS as the “minimal requirement” for self-enforceability of coalition deviations. On the other hand, the IR is weaker than the NS, and hence it does not satisfy this minimal requirement. If we would like to single out a particular Nash equilibrium from multiple Nash equilibria, we must apply self-enforcing conditions, which are mathematically definable but may be unjustifiable as “natural" coalitional behavior in economic meaning.

Appendix: Proofs

Proof of Proposition 1

(1), (2), (3), and (5) are immediate from the definitions of equilibria.

(4) Let $D \in \mathcal{C}$ and let $s \in E^{\Gamma}_{(\mathcal{C},\mathcal{R}_{\mathcal{C}})}$. First, suppose that $R^s_D = E_N^{\Gamma|s_{-D}}$. By the definition of s-coalition-proof Nash equilibrium, the set of s-self-enforcing deviations of D is a subset of $E_N^{\Gamma|s_{-D}}$. Second, suppose that $R^s_D = \{s'_D \in S_D |\, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\setminus\{i\}}, s_{-D})$ for each $i \in D\}$. We then note that $E_N^{\Gamma|s_{-D}} \subseteq R^s_D$ and the set of s-self-enforcing deviations of D is a subset of $E_N^{\Gamma|s_{-D}}$. Hence, in any case, $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$-coalitional equilibrium is robust to the self-enforcing deviations. ■

Proof of Fact 1

First, as a preparation for proof of Fact 1, we show Claim 1.

Claim 1 If $S_i = [0, 5.6]$ for each $i \in N$, then $10 - s_i - s_j$, $10 - s_j$, and $\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}$ are decreasing in s_j.

Proof of Claim 1. Clearly, $10 - s_i - s_j$ and $10 - s_j$ are decreasing in s_j. Differentiating $\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}$ in s_j, we

[9] The coalition-proof Nash equilibrium under weak Pareto domination is defined by replacing strong Pareto dominance of s-coalition-proof Nash equilibria with weak Pareto dominance. See also Corollary 2 in Quartieri and Shinohara (2015).

[10] For further information, all Nash equilibria in this example are *strict* Nash equilibria, which are also trembling-hand-perfect Nash equilibria (Selten, 1975; Okada, 1981). Hence, trembling perfection does not single out a particular Nash equilibrium either.

have

$$\frac{\partial}{\partial s_j}\left(\frac{2}{9-s_j}s_i+\frac{79-18s_j+s_j^2}{9-s_j}\right)=\frac{2s_i-s_j^2+18s_j-83}{(9-s_j)^2}.$$

Since $S_i = S_j = [0, 5.6]$, then $2s_i - s_j^2 + 18s_j - 83$ is maximized at $(s_i, s_j) = (5.6, 5.6)$ and the maximum value is -2.36. Thus, $\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}$ is also decreasing in s_j. **(End of Proof of Claim 1)**

We first verify that this game satisfies σ-DE. Let $s_i = x \in S_i$ and let $s_j', s_j'' \in S_j$ be such that $s_j'' < s_j'$. We show that $u_i(x, s_j'') \geq u_i(x, s_j')$. Note that $1 < \min\{10 - s_j', 5.6\} \leq \min\{10 - s_j'', 5.6\}$ and the last inequality holds with equality if $s_j' \leq 4.4$.

By Claim 1, if $x \in [0, 1] \cup [1, \min\{10 - s_j', 5.6\}] \cup (\min\{10 - s_j'', 5.6\}, 5.6]$, then $u_i(x, s_j') < u_j(x, s_j'')$ because

$$u_i(x,z)=\begin{cases}10-x-z & \text{if } x\in[0,1]\\ \frac{2}{9-z}x+\frac{79-18z+z^2}{9-z} & \text{if } x\in[1,\min\{10-s_j',5.6\}]\\ 10-z & \text{if } x\in(\min\{10-s_j'',5.6\},5.6]\end{cases}$$

for each $z \in \{s_j', s_j''\}$. If $x \in \left(\min\{10 - s_j', 5.6\},\ \min\{10 - s_j'', 5.6\}\right]$, then $u_i(x, s_j') = 10 - s_j'$ and $u_j(x, s_j'') = \frac{2}{9-s_j''}x + \frac{79-18s_j''+(s_j'')^2}{9-s_j''}$. Denote $s_j' = s_j'' + d$, where $d > 0$. Then, we have

$$\begin{aligned}u_j(x,s_j'')-u_j(x,s_j') &= \frac{-11+s_j''+2x+d(9-s_j'')}{9-s_j''}\\ &> \frac{-11+s_j''+2\min\{10-s_j',5.6\}+d(9-s_j'')}{9-s_j''}\\ &= \begin{cases}\frac{9-s_j''+d(7-s_j'')}{9-s_j''}>0 & \text{if } \min\{10-s_j',5.6\}=10-s_j'\\ \frac{0.2+s_j''+d(9-s_j'')}{9-s_j''}>0 & \text{otherwise}\end{cases}\end{aligned}$$

because $s_j'' \leq 5.6$. In conclusion, this game satisfies σ-DE.

We secondly verify that this game satisfies σ-SS. Let $i, j \in N$ be such that $i \neq j$ and $s \in S_N$. First, if $s_j \in (4.4, 5.6]$, then $u_i(s)$ is maximized at $s_i = 10 - s_j$ as we can see in Figure 1. Second, if $s_j \in [0, 4.4]$, then $u_i(s)$ is *locally* maximized at $s_i = 0,\ 5.6$ and $u_i(5.6, s_j) - u_i(0, s_j) = \frac{0.2+s_j}{9-s_j} > 0$. Therefore,

$$b_i(s)=\begin{cases}\{10-s_j\} & \text{if } s_j\in(4.4,5.6]\\ \{5.6\} & \text{otherwise}\end{cases}. \tag{2}$$

Clearly, this is a game with σ-SS.

We can also verify that this is not a game with σ-SCP. We have that $u_1(5.2, 4.5) = u_1(5.4, 4.5) = 5.5$ and $u_1(5.2, 5) = u_1(5.4, 5) = 5$; hence, $u_1(5.2, 4.5) - u_1(5.4, 4.5) = u_1(5.2, 5) - u_1(5.4, 5) = 0$, which implies that this game does not satisfy σ-SS. ■

Proof of Fact 2

By (2),

$$E_N^\Gamma = \{(s_1, s_2) : s_1 + s_2 = 10 \text{ and } 4.4 \leq s_i \leq 5.6 \text{ for each } i \in N\}.$$

First, we verify $e^* = (5, 5) \in E_N^\Gamma \backslash E_{(C,\mathcal{R}_C)}^\Gamma$. The payoff to all $i \in N$ at e^* is $u_i(e^*) = 6$. If the two players deviate from e^* to $e = (0, 0)$, then $u_i(e) = 10$ for all $i \in N$. If player i switches back to $e_i^* = 5$ given e_j for $j \neq i$, then

$u_i(e_i^*, e_j) = \frac{89}{9}$. Therefore, no player i switches back to the original strategy e_i^*.

Second, we verify that $e^{**} = (4.4, 5.6) \in E^{\Gamma}_{(C,\mathcal{R}_C)}$. At e^{**}, $u_1(e^{**}) = 5.4$ and $u_2(e^{**}) = 6.6$. Let $\tilde{s} \in S_N$ be deviating strategies from e^{**} such that $u_i(\tilde{s}) > u_i(e^{**})$ for each $i \in N$. Since this is a game with σ-DE, then $e_i^{**} > \tilde{s}_i$ for each $i \in N$ by Lemma 1. We then have

$$\tilde{s}_i < e_i^{**} = 10 - e_j^{**} < 10 - \tilde{s}_j \text{ for all } i, j \in N \text{ such that } i \neq j. \tag{3}$$

Claim 2 If there exist distinct $i, j \in N$ such that $\tilde{s}_i \in \left[1, \min\{10 - \tilde{s}_j, 5.6\}\right]$, then $u_i(e_i^{**}, \tilde{s}_j) > u_i(\tilde{s})$.

Proof of Claim 2. By (3), $e_i^{**} \in [1, \min\{10 - \tilde{s}_j, 5.6\}]$. Hence, for each $x \in \{e_i^{**}, \tilde{s}_i\}$,

$$u_i(x, \tilde{s}_j) = \frac{1}{9 - \tilde{s}_j}\left(2x + 79 - 18\tilde{s}_j + (\tilde{s}_j)^2\right)$$

and $e_i^{**} > \tilde{s}_i$ implies $u_i(e_i^{**}, \tilde{s}_j) > u_j(\tilde{s})$. **(End of Proof of Claim 2)**

Claim 3 If there exist distinct $i, j \in N$ such that $\tilde{s}_i \in \left(\min\{10 - \tilde{s}_j, 5.6\}, 5.6\right]$, then $u_i(e_i^{**}, \tilde{s}_j) > u_i(\tilde{s})$.

Proof of Claim 3. Since $e_1^{**} = 4.4 > \tilde{s}_1$, then it is impossible that $i = 2$ and $j = 1$. (If $i = 2$ and $j = 1$, then $\left(\min\{10 - \tilde{s}_j, 5.6\}, 5.6\right]$ is empty.) We consider the case of $i = 1$ and $j = 2$. In this case, note that $\tilde{s}_2 \geq 4.4$. Since $\tilde{s}_2 \leq 5.6$, then $\min\{10 - \tilde{s}_2, 5.6\} \geq 4.4$. Since $e_1^{**} = 4.4$, then $e_1^{**} \in [1, \min\{10 - \tilde{s}_2, 5.6\}]$. Hence,

$$\begin{aligned} u_1(\tilde{s}) - u_1(e_1^{**}, \tilde{s}_2) &= 10 - \tilde{s}_2 - \left(\frac{87.8 - 18\tilde{s}_2 + (\tilde{s}_2)^2}{9 - \tilde{s}_2}\right) \\ &= \frac{2.2 - \tilde{s}_2}{9 - \tilde{s}_2} < 0. \end{aligned}$$

(End of Proof of Claim 3)

Claim 4 If $\tilde{s} \in [0, 1]^2$, then $u_2(\tilde{s}_1, e_2^{**}) > u_2(\tilde{s})$.

Proof of Claim 4. By (1), since $\tilde{s} \in [0, 1]^2$, then $u_i(\tilde{s}) \in [8, 10]$ for each $i \in N$. We have

$$\begin{aligned} u_2(\tilde{s}) - u_2(\tilde{s}_1, e_2^{**}) &= 10 - \tilde{s}_1 - \tilde{s}_2 - \left(\frac{90.2 - 18\tilde{s}_1 + (\tilde{s}_1)^2}{9 - \tilde{s}_1}\right) \\ &= -\frac{0.2 + \tilde{s}_1 + \tilde{s}_2(9 - \tilde{s}_1)}{9 - \tilde{s}_1} < 0 \end{aligned}$$

because $\tilde{s}_1 \leq 1$. **(End of Proof of Claim 4)**

By Claims 2 to 4, for each improving deviation $\tilde{s}$, there is at least one player $k \in N$ that switches back to e_k^{**}. Therefore, $e^{**} \in E^{\Gamma}_{(C,\mathcal{R}_C)}$.

In conclusion, $\emptyset \neq E^{\Gamma}_{(C,\mathcal{R}_C)} \subsetneq E^{\Gamma}_N$. ■

References

[1] Aumann, R. (1959) Acceptable points in general cooperative n-person games. In: Tucker, A.W., Luce, D.R. (eds) Contributions to the theory of games IV. Princeton University Press, Princeton, 287-324.

[2] Bernheim, D., Peleg, B., Whinston, M. (1987) Coalition-proof Nash equilibria I: Concepts. Journal of Economic Theory, vol. 42, 1-12.

[3] Ichiishi T. (1981) A social coalitional equilibrium existence lemma. Econometrica, vol. 49, 369-377.

[4] Kaplan, G. (1992) Sophisticated outcomes and coalitional stability. M.Sc. thesis, Department of Statistics, Tel Aviv University.

[5] Laraki, R (2009) Coalitional equilibria of strategic games. Ecole Polytechnique.

[6] Milgrom, P., Roberts, J. (1994) Strongly coalition-proof equilibria in games with strategic complementarities. Stanford University.

[7] Milgrom, P., Roberts, J. (1996) Coalition-proofness and correlation with arbitrary communication possibilities. Games and Economic Behavior, vol. 17, 113-128.

[8] Okada, A. (1982) On stability of perfect equilibrium points. International Journal of Game Theory, vol.10, 67-73.

[9] Quartieri, F. (2013) Coalition-proofness under weak and strong Pareto dominance. Social Choice and Welfare, vol.40, 553-579.

[10] Quartieri, F., Shinohara, R. (2015) Coalition-proofness in a class of games with strategic substitutes. International Journal of Game Theory, vol. 44, 785-813.

[11] Rozenfeld, O., Tennenholtz, M. (2010) Near-strong equilibria in network creation games. Internet and Network Economics, Lecture Notes in Computer Science, vol. 6484, 339-353.

[12] Selten, R. (1975) Reexamination of the perfectness concept for equilibrium points in extensive games. International Journal of Game Theory, vol. 4, 25-55.

[13] Shinohara, R. (2005) Coalition-proofness and dominance relations. Economics Letters, vol. 89, 74-179.

[14] Shinohara, R. (2010) Coalition-proof Nash equilibrium of aggregative games. Available at http://ryusukeshinohara.ehoh.net/wp/agg.pdf.

[15] Shinohara, R. (2019) Undominated coalition-proof Nash equilibria in quasi-supermodular games with monotonic externalities. Economics Letters, vol. 176, 86-89.

[16] Yi, S. (1999) On the coalition-proofness of the Pareto frontier of the set of Nash equilibria. Games and Economic Behavior, vol. 26, 353-364.

[17] Zhao, J. (1992) The hybrid solution of an N-person game. Games and Economic Behavior, vol. 4, 145-160.

Journal of International Economic Studies (2020), No.34, 23–52

Undertaking nonharmful or harmful public projects through unit-by-unit contribution: Coordination and Pareto efficiency

Ryusuke Shinohara*

Abstract

We examine in detail the implementation of a project that is nonharmful for all agents as well as a project that is harmful for some agents through a *unit-by-unit contribution mechanism.* For a project that is nonharmful for all agents, efficient implementation is supported at one regular Nash equilibrium and several refined Nash equilibria that are stable against coalition deviations. In this sense, this mechanism works well. On the other hand, when the project is harmful for some agents, this mechanism may not have a Nash equilibrium with efficient implementation of the project. Even when such a Nash equilibrium exists, it may not be selected by any of the refined Nash equilibria. Thus, in this case, this mechanism does not work. Our result shows that the merit of the unit-by-unit contribution mechanism reported in the literature is partially extensible to the implementation of a public project.

Keywords: Public project; Unit-by-unit contribution; Pareto efficiency; Strong Nash equilibria; Coalition-proof Nash equilibria.

JEL Classification: **C72, D62, D74, H41.**

1 Introduction

We consider a public project implementation through a *unit-by-unit contribution mechanism.* We investigate in detail the implementation of a project that is nonharmful for all agents as well as a project that is harmful for some agents. We examine under what conditions the project is undertaken Pareto-efficiently through the unit-by-unit contribution mechanism.

The unit-by-unit contribution mechanism is introduced to provide a discrete pure public good in integer units. As in a standard case of public-good provision in nonnegative real numbers, voluntary public-good provision in nonnegative integer units suffers from the free-rider problem, so that the public good is not supplied Pareto-efficiently.[1] One of the solutions to this problem is to construct public-good mechanisms. To solve the free-rider problem of an integer-unit public good, Bagnoli and Lipman (1989) introduce a unit-by-unit contribution mechanism. Later, Brânzei et al. (2005) introduced another mechanism, which is a little different from, but essentially the same as, Bagnoli and Lipman's

*Department of Economics, Hosei University, 4342 Aihara-machi, Machida, Tokyo, 194-0298, Japan. Tel: (81)-42-783-2534. Fax: (81)-42-783-2611. E-mail: ryusukes@hosei.ac.jp

[1]For the voluntary provision of an integer-unit public good, see, for example, Bagnoli and Lipman (1989, p.591, last paragraph), Gradstein and Nitzan (1990), and Shinohara (2009).

(1989) mechanism[2] and applied it to a public-good problem that is different from the Bagnoli and Lipman (1989) problem. Their mechanisms for solving the problem are based on the idea that the level of public-good provision is decided through a "unit-by-unit" process. In their mechanisms, agents are asked to make marginal contributions to every one-unit increase in the public good. Based on the contributions, starting from the first unit of the good, the quantity increases by one unit as long as the sum of the marginal contributions to a one-unit increase covers its marginal cost. Bagnoli and Lipman (1989) and Brânzei et al. (2005) show that the unit-by-unit contribution mechanism has a Nash equilibrium at which the public good is provided Pareto-efficiently. Moreover, they show that although this mechanism may have other Nash equilibria at which the public good is provided Pareto-inefficiently, some refinements of Nash equilibria single out the Nash equilibria with efficient provision of the public good. In this sense, the mechanism solves the free-rider problem of the provision of an integer-unit public good.[3]

We could say that this mechanism is based on a "simple" rule: whether the public good increases by one unit depends only on the relationship between the marginal contributions to and the marginal cost of this increase and the payment from each agent is the sum of her announced marginal contributions to each unit. Moreover, we could say that this mechanism is "suitable" in the provision of an integer-unit public good because it utilizes a discrete structure of an integer-unit public good. Because of this simplicity and suitability, it seems to have some applicability to the implementation of public projects in the real world. Hence, it would be important to know how this mechanism works in the provision of various public projects.

However, this mechanism has been tested under limited situations in the literature. Bagnoli and Lipman (1989) and Brânzei et al. (2005) assume that agents have a quasi-linear utility function with respect to a private good and benefits from a public good are measured in terms of the private good. Bagnoli and Lipman (1989) assume that agents' benefit functions from the public good are increasing and strictly concave in level, which are seemingly standard conditions for public good provision. On the other hand, Brânzei et al. (2005) assume that each agent has a threshold level of the public good and receives a positive constant benefit if and only if the public good is provided at the threshold level or higher. How this mechanism works has not been clarified in the implementation of public projects that cannot be captured by those benefit structures.

Moreover, when it comes to public projects in the real world, they are sometimes harmful in the sense that raising the level of a public project may decrease someone's benefits. For example, consider the construction of a high-speed railway (HSR) network such as the *Shinkansen* bullet-train projects in Japan. This project connects Tokyo (the capital city) to the peripheral cities with HSR networks, which have been extended sequentially.[4] It is said that this extension has two sides: it may stimulate the local economies since tourism is promoted and some companies in the capital city establish branch offices in the local cities. On the other hand, it may create disadvantages such as outflow of population from local cities. In reality, these positive and negative sides would determine the benefits to peripheral

[2] See a detailed explanation of this point in Section 2.

[3] To be precise, Bagnoli and Lipman (1989) use a refinement of trembling-perfect Nash equilibria and Brânzei et al. (2005) use a strong Nash equilibrium (Aumann, 1959). Their refinement concepts are completely different. They prove that payoffs attained at those refined Nash equilibria coincide with the core of a cooperative game. We also use several refinements of Nash equilibria based on coalition formation, including the strong Nash equilibrium.

[4] For instance, Tokyo and Nagano City (a city about 220 km away from Tokyo) were connected by the HSR network in 1997. This network was extended to Kanazawa City (a city about 450 km away from Tokyo) in 2015.

cities. Some empirical studies show that extension of the HSR network does not necessarily benefit peripheral cities.[5] If we interpret this extension as an increase in the project level, some agents might lose their benefits from the project by the increase. This shows public project effects that cannot be captured by the benefit structures of earlier studies. When examining the applicability of the unit-by-unit contribution mechanism to the implementation of real-world public projects, we need to consider the case in which a public project is harmful for some agents. However, this has not been considered in the literature.

In order to examine applicability of the unit-by-unit contribution mechanism, we need to introduce a framework that can capture as many public projects as possible. We introduce two types of public projects—one is "nonharmful" for all agents and the other is sometimes "harmful" for some agents—and examine the implementation of each public project through the unit-by-unit contribution mechanism. Our aim is to clarify to what extent this mechanism achieves efficient public project implementation in each case .

Firstly, a project is defined to be *nonharmful for all agents* if their benefit functions from the project are *weakly increasing* in the level of the project. The weakly increasing benefit functions are worth analyzing because they are a generalization of the benefit functions of Bagnoli and Lipman (1989) and Brânzei et al. (2005). We show that the unit-by-unit contribution mechanism always has a Nash equilibrium at which the nonharmful public project is undertaken Pareto-efficiently, although it may have a Nash equilibrium at which the project is done inefficiently. We further prove that with and without monetary transfers, the set of Nash equilibria with efficient project implementation coincides with the set of strong Nash equilibria and the set of coalition-proof Nash equilibria (Bernheim et al., 1987) (Theorem 1). These results show that although multiple public project levels may be supported at the Nash equilibria, only Nash equilibria with efficient project implementation are supported by various Nash equilibrium refinements that are robust to coalition deviations. Theorem 1 supplements the results of earlier studies as follows: Firstly, in the earlier studies, the weakly increasing property of the benefit functions is a key factor in the mechanism of efficient public good provision at a Nash equilibrium. Second, the Nash equilibria for efficient projects are much more robust to coalition deviations than are shown by Brânzei et al. (2005) because they test only a strong Nash equilibrium without transfers.

Secondly, a project is considered *harmful for some agents* if their benefit functions from the project are not weakly increasing in level. We additionally impose *weak concavity* on the benefit functions of all agents for tractability. We show that the unit-by-unit contribution mechanism does not always work well in the implementation of a harmful project. Unlike in nonharmful projects, this mechanism does not always have a Nash equilibrium with efficient public project implementation. Moreover, this mechanism may have a Nash equilibrium at which the project is undertaken at a level exceeding the efficient level. We establish necessary and sufficient conditions for a Nash equilibrium with implementation of the project at or over the efficient level (see Propositions 1 and 2). As for nonharmful projects, these conditions lead to the possibility of multiple Nash equilibria with both efficient and inefficient implementation of the public project. We then examine the strong Nash equilibrium and coalition-proof Nash equilibrium to clarify the level of project implementation—the efficient level or the over-implementation level—that is robust to coalition deviations. We observe that these refined

[5]For a Japanese case, see, for example, Sasaki et al. (1997). Similar effects have been observed from the extension of HSR networks in European countries. See, for example, Ureña et al. (2009).

Nash equilibria do not always select Nash equilibria with efficient project implementation. Firstly, we find that the mechanism may not have strong Nash equilibria with or without transfers. Secondly, although coalition-proof Nash equilibria with and without transfers do exist, they do not always single out Nash equilibria with efficient project implementation. Coalition-proof Nash equilibria single out Nash equilibria with efficient project implementation if and only if there is a Nash equilibrium with efficient project implementation, and no other Nash equilibria with over-implementation (Theorem 3). Finally, we introduce a reasonably large class of modified unit-by-unit contribution mechanisms and investigate whether this modified mechanism achieves the efficient undertaking of harmful public projects. We show that no mechanism in this class implements an efficiency project in Nash equilibria (Proposition 5).

In conclusion, when the project is nonharmful for all agents, the unit-by-unit contribution mechanism works well since it only achieves an efficient project at various refined Nash equilibria. On the other hand, when the project is harmful for some agents, the mechanism does not necessarily work since it may not have a Nash equilibrium with an efficient project. Furthermore, even if it has such a Nash equilibrium, none of the refined Nash equilibria based on coalition deviations considered in this paper singles it out. Thus, whether the unit-by-unit contribution mechanism works depends on the properties of the project. The merit of the unit-by-unit contribution mechanism reported in the literature is extensible to the implementation of a nonharmful project, but only partially extensible to that of a harmful public project. If we aim to achieve efficient project implementation under general benefit structures at various refined Nash equilibria based on coalition deviations, we need to consider another class of modified unit-by-unit contribution mechanisms or construct new mechanisms.

Finally, we mention some related studies. Our conditions on benefit functions from a public project could be compared with several classes of benefit functions of Laussel and Le Breton (2001). In our model, if all agents have weakly increasing benefit functions, then the *comonotonicity* condition of Laussel and Le Breton (2001) holds. Otherwise, it does not. *The two-sided* property of Laussel and Le Breton (2001), another condition of benefit structures, does not hold in our model.[6] Thus, our benefit function conditions cannot be fully captured by the Laussel and Le Breton (2001) classes of benefit functions. In this sense, we analyze a new class of benefit functions. However, note that Laussel and Le Breton (2001) work on the common agency game, which is different from our unit-by-unit contribution game because ours does not have a profit-maximizing common agency to implement public projects. There seems to be little significance in comparing to compare their results with ours.

To the best of my knowledge, apart from Bagnoli and Lipman (1989) and Brânzei et al. (2005), only Yu (2005) proposes a mechanism, which is completely different from the unit-by-unit contribution mechanism, for provision of an integer-unit pure public good. Her two-stage mechanism implements any one of the allocations in the core in an undominated subgame-perfect Nash equilibrium. A *voluntary participation problem*, pointed out by Saijo and Yamato (1999), can be captured as another free-rider problem of public good provision related to the participation decision in a public good mechanism. Nishimura and Shinohara (2013) propose a multi-stage mechanism, called a unit-by-unit *participation* mechanism, and show that the idea of a unit-by-unit process can mitigate this problem. Although the unit-by-unit participation mechanism and our mechanism are totally different, Nishimura and Shinohara (2013) do not explore the extensibility of the merit of the unit-by-unit participation mechanism

[6]For the definitions of comonotonicity and two-sidedness, see Laussel and Le Breton (2001).

to the implementation of harmful or nonharmful projects. Shinohara (2014) investigates a voluntary participation problem in which agents have the same benefit functions as those of Brânzei et al. (2005). Shinohara (2014) does not study this extensibility, either.

The paper is organized as follows: Section 2 introduces the model and equilibrium concepts. Section 3 presents the results for nonharmful projects. Section 4 provides the results for harmful projects. Section 5 concludes the study. The proofs of the propositions in Sections 3 and 4 are collated in the appendices.

2 The model

Consider an economy in which agents undertake a public project through contribution of a private good (money). The level of the public project is assumed to take a nonnegative integer. Let $\mathcal{Y} = \{0, 1, \dots, \bar{y}\}$ be the set of project levels, where $\bar{y}$ is an integer greater than or equal to one, and the finite upper bound of the public project level. Let $c : \mathcal{Y} \to \mathbb{R}_+$ be a cost function of the project such that $c(0) = 0$. For all $y, y' \in \mathcal{Y}$ such that $y \geq y'$, let $\Delta c(y, y') \equiv c(y) - c(y')$ be the additional (marginal) cost from y' to y units. We assume that c is an increasing and weakly convex function in $\mathcal{Y}$: that is,

$$\begin{aligned} &\Delta c(y+1, y) > 0 \text{ for all } y \in \mathcal{Y} \\ \text{and } &\Delta c(y+1, y) \geq \Delta c(y'+1, y') \text{ for all } y, y' \in \mathcal{Y} \text{ such that } y > y'. \end{aligned} \tag{1}$$

Let $N = \{1, \dots, n\}$ be the set of agents such that n is a finite integer and $n \geq 1$. Each agent $i \in N$ has a quasi-linear utility function $U_i : \mathcal{Y} \times \mathbb{R}_+ \to \mathbb{R}$ such that $U_i(y, t_i) = u_i(y) - t_i$, in which $u_i : \mathcal{Y} \to \mathbb{R}$ is agent i's benefit function from the project with $u_i(0) = 0$ and t_i is i's private-good contribution to the project. For all $y, y' \in \mathcal{Y}$ such that $y \geq y'$, let $\Delta u_i(y, y') \equiv u_i(y) - u_i(y')$ be agent i's additional (marginal) benefit from the increase from y' to y units.

We assume that the project has a "public-good nature"; that is, every agent benefits from the same project level, irrespective of his contribution. However, we do not always assume that the project is a public "good." We allow the case in which a higher project level may harm some agents, while it benefits others. In the subsequent sections, we impose additional conditions on u_i, which determine the project character. Note that in our model, agents who benefit from a higher project level, if any, want to free-ride others' contribution. That is, the free-rider problem does matter.

We identify an *economy* by a list $[N, (u_i)_{i \in N}, c]$. For each economy, the existence of the Pareto-efficient level for a project is trivial since $\mathcal{Y}$ is a finite set. For analytical simplicity, we assume that $y^* \in \mathcal{Y}$ is a unique efficient project level, where y^* is positive;[7] that is, $\{y^*\} = \arg\max_{y \in \mathcal{Y}} \sum_{i \in N} u_i(y) - c(y)$.[8] We also assume that for all coalitions $D \subseteq N$, $\arg\max_{y \in \mathcal{Y}} \sum_{j \in D} u_j(y) - c(y)$ is a singleton. For all $D \subseteq N$, let $Y(D) \in \mathcal{Y}$ be a *stand-alone level of the project for* D such that $\{Y(D)\} = \arg\max_{y \in \mathcal{Y}} \sum_{j \in D} u_j(y) - c(y)$. We do not assume that $Y(D)$ is positive for all $D \subsetneq N$. Let $Y^{\max} \equiv \max_{D \subseteq N} Y(D)$. The assumption of a unique stand-alone level for each coalition is used only in Section 4.

[7] The subsequent analysis is applicable to the trivial case of $y^* = 0$.

[8] The notion of efficiency in this study is based on transferable resources. That is, if we denote w_i and x_i as the initial endowment of the private good and the consumption of it for all $i \in N$, respectively, then the resource constraint is $\sum_{i \in N} w_i \geq \sum_{i \in N} x_i + c(y)$. This constraint is rewritten as $\sum_{i \in N} t_i \geq c(y)$ because $t_i = w_i - x_i$ for all $i \in N$. If we further assume that w_i is sufficiently large for all $i \in N$, then the allocation is efficient if and only if it maximizes the total surplus. Regarding this, see, for example, Silvestre (2012).

We immediately obtain Lemma 1 from the uniqueness of the efficient level y^*.

Lemma 1 For all $y \in \mathcal{Y}$, $\sum_{j \in N} \Delta u_j(y^*, y) > \Delta c(y^*, y)$ if $y^* > y$ and $\sum_{j \in N} \Delta u_j(y, y^*) < \Delta c(y, y^*)$ if $y^* < y$.

Proof. By the efficiency and the uniqueness of y^*, $\sum_{j \in N} u_j(y^*) - c(y^*) > \sum_{j \in N} u_j(y) - c(y)$ for all $y \in \mathcal{Y} \backslash \{y^*\}$, which implies the conditions in the statement. ■

We focus on the undertaking of a public project through a *unit-by-unit contribution mechanism*, which is the same as the mechanism of Brânzei et al. (2005). In this mechanism, each agent $i \in N$ simultaneously chooses a vector of marginal contributions to each one-unit increase of the project. Let $\sigma_i \equiv (\sigma_i^y)_{y \in \mathcal{Y} \backslash \{0\}} \in \mathbb{R}_+^{\bar{y}}$ be a typical vector of marginal contributions chosen by agent i, in which $\sigma_i^y \in \mathbb{R}_+$ is a marginal contribution from i to the marginal production from $y - 1$ to y units. The project level is determined as follows: $y \in \mathcal{Y} \backslash \{0\}$ units of the project are undertaken at $\sigma = (\sigma_i)_{i \in N}$ if and only if (i) for all units of $\hat{y}$, which is less than or equal to y, the sum of contributions to the $\hat{y}$-th unit of the project, $\sum_{i \in N} \sigma_i^{\hat{y}}$, covers the marginal cost of that unit, $\Delta c(\hat{y}, \hat{y} - 1)$, and (ii) the sum of contributions to $y + 1$-th unit, $\sum_{i \in N} \sigma_i^{y+1}$, falls short of the marginal cost $\Delta c(y + 1, y)$. If the marginal cost of the first unit is not covered by the sum of contributions to that unit, then the project level is zero. Formally, for each $\sigma = (\sigma_i)_{i \in N} \in \mathbb{R}_+^{n\bar{y}}$, let $y(\sigma)$ be the public project level at σ such that

$$y(\sigma) \equiv \max \left\{ y \in \mathcal{Y} \;\middle|\; \sum_{i \in N} \sigma_i^{\hat{y}} \geq \Delta c(\hat{y}, \hat{y} - 1) \text{ for all } \hat{y} \in \mathcal{Y} \text{ such that } \hat{y} \leq y. \right\}, \tag{2}$$

where we define $\sigma_i^0 \equiv 0$ for all $i \in N$ and $c(0) - c(0 - 1) \equiv 0$ for consistency. For all $\sigma \in \mathbb{R}_+^{n\bar{y}}$, each agent i pays $\sum_{y \in \mathcal{Y} \backslash \{0\}} \sigma_i^y$. In this mechanism, the marginal contribution to some unit is never refunded even though the project is not undertaken at that unit. However, as we will see later, the contribution is never wasted at every Nash equilibrium.

The mechanism accompanied with $(U_i)_{i \in N}$ constitutes a strategic-form game $\Gamma = [N, (S_i, V_i)_{i \in N}]$, in which $S_i \equiv \mathbb{R}_+^{\bar{y}}$ is the set of strategies for $i \in N$ and $V_i : \prod_{j \in N} S_j \longrightarrow \mathbb{R}$ is agent i's payoff function, depending on strategies such that $\sigma \in \prod_{j \in N} S_j \longmapsto V_i(\sigma) \equiv U_i(y(\sigma), \sum_{y \in \mathcal{Y} \backslash \{0\}} \sigma_i^y) \in \mathbb{R}$. Hereafter, we call Γ a *unit-by-unit contribution game*. The unit-by-unit contribution game is a complete information game.

Bagnoli and Lipman (1989) introduce a *multi-stage* unit-by-unit contribution mechanism. It starts with the decision on whether to provide the first unit of the project. In the first stage, the agents contribute to the first unit of the project. If the sum of contributions to the first unit covers the marginal cost for that unit, the first unit is provided, and the agents go to the second stage. Otherwise, the first unit is not provided, and the mechanism ends. If the agents go to the second stage, it is decided in the same way whether or not to provide a second unit. The second unit is provided, and the agents go to the third stage if and only if the sum of contributions to the second unit covers the marginal cost for that unit. This continues till the sum of contributions to a one-unit increase falls short of the marginal cost for that increase. We consider the mechanisms of Brânzei et al. (2005) and Bagnoli and Lipman (1989) as essentially the same because the decision on a one-unit increase of the public good is based on the relationship between the marginal contribution and the marginal cost for that unit. In this paper, we analyze the mechanism based on a simultaneous game.

We introduce equilibrium concepts for the unit-by-unit contribution game. Our analysis is restricted to pure strategies. The Nash equilibrium is defined as usual.

For each $D \subseteq N$, denote a strategy profile for D by $\sigma_D \in \prod_{j \in D} S_j$. We simply write $\sigma_N = \sigma$. A *strong Nash equilibrium* (Aumann, 1959) is a Nash equilibrium that is stable against all possible coalition deviations.

Definition 1 Strategy profile $\sigma \in \prod_{j \in N} S_j$ is a *strong Nash equilibrium* of Γ if there is no $D \subseteq N$ and $\sigma'_D \in \prod_{j \in D} S_j$ such that $V_j(\sigma) < V_j(\sigma'_D, \sigma_{N \setminus D})$ for all $j \in D$.

A *coalition-proof Nash equilibrium* (Bernheim et al., 1987) is also an equilibrium based on stability against coordinated strategies. Unlike the strong Nash equilibrium, the coalition-proof Nash equilibrium is limited to "self-enforcing" coalitional deviations. This equilibrium is based on the notion of a restricted game. For all $D \subsetneq N$ and all $\sigma_{N \setminus D} \in \prod_{j \in N \setminus D} S_j$, $\Gamma|\sigma_{N \setminus D}$ is a *restricted game of* Γ *at* $(D, \sigma_{N \setminus D})$ in which the agents in D plays Γ, taking as given that the other agents choose $\sigma_{N \setminus D}$; that is, $\Gamma|\sigma_{N \setminus D}$ is a list $[D, (S_i, \tilde{V}_i)_{i \in D}]$ in which D is a set of players for each $i \in D$, $S_i = \mathbb{R}_+^{\bar{y}}$ is i's strategy set, and $\tilde{V}_i$ is the payoff function of i such that $\tilde{\sigma}_D \in \prod_{i \in D} S_j \longmapsto \tilde{V}_i(\tilde{\sigma}_D) \equiv V_i(\tilde{\sigma}_D, \sigma_{N \setminus D}) \in \mathbb{R}$.

Definition 2 A coalition-proof Nash equilibrium $\sigma \in \prod_{j \in N} S_j$ is defined inductively with respect to the number of agents $n \geq 1$. Suppose that $n = 1$. Then, $\sigma \in \prod_{j \in N} S_j$ is a *coalition-proof Nash equilibrium of* Γ if σ is a Nash equilibrium of Γ.

Suppose that $n \geq 2$ and suppose that a coalition-proof Nash equilibrium has been defined for all games with fewer than n agents. $\sigma \in \prod_{j \in N} S_j$ is *self-enforcing in* Γ if it is a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N \setminus D}$ for all nonempty $D \subsetneq N$. $\sigma \in \prod_{j \in N} S_j$ is a *coalition-proof Nash equilibrium of* Γ if it is self-enforcing in Γ and there is no other self-enforcing strategies $\sigma' \in \prod_{j \in N} S_j$ in Γ such that $V_j(\sigma) < V_j(\sigma')$ for all $j \in N$.

The self-enforcing property of coalition-proof Nash equilibria restricts possible coalition deviations, and hence the set of strong Nash equilibria is always a subset of the set of coalition-proof Nash equilibria.

Since we assume that agents have quasi-linear utility functions, it would be appropriate to consider coalition deviations through monetary transfers. Consider a situation in which a coalition $D \subseteq N$ deviates and each of its members freely sends transfers to other members. Let $i \in D$ and $\tau_i \in \mathbb{R}$ be a *net* transfer to agent i from the others: τ_i is equal to the transfers i sends minus the transfers she receives. There is no outside transfer resource; that is, $\sum_{i \in D} \tau_i = 0$. Based on these kinds of transfers, we redefine the strong Nash and coalition-proof Nash equilibria.

Definition 3 Strategy profile $\sigma \in \prod_{j \in N} S_j$ is a *strong Nash equilibrium with transfers* of Γ if there is no $D \subseteq N$, $\sigma'_D \in \prod_{j \in D} S_j$ and $(\tau_j)_{j \in D} \in \mathbb{R}^{|D|}$ such that $\sum_{j \in D} \tau_j = 0$ and $V_i(\sigma) < V_i(\sigma'_D, \sigma_{N \setminus D}) + \tau_i$ for all $i \in D$.

Note that σ is a strong Nash equilibrium with transfers if and only if there is no $D \subseteq N$ and $\sigma'_D \in \prod_{j \in D} S_j$ such that $\sum_{j \in D} V_j(\sigma) < \sum_{j \in D} V_j(\sigma'_D, \sigma_{N \setminus D})$. That is, no coalition can deviate from a strong Nash equilibrium with transfers so as to increase the sum of payoffs of its members.

Definition 4 A coalition-proof Nash equilibrium with transfers $\sigma \in \prod_{j\in N} S_j$ is defined inductively with respect to the number of agents $n \geq 1$. Suppose that $n = 1$. Then, $\sigma \in \prod_{j\in N} S_j$ is a *coalition-proof Nash equilibrium with transfers of* Γ if σ is a Nash equilibrium of Γ.

Suppose that $n \geq 2$ and suppose that a coalition-proof Nash equilibrium with transfers has been defined for all games with fewer than n agents. $\sigma \in \prod_{j\in N} S_j$ is *self-enforcing with transfers in* Γ if it is a coalition-proof Nash equilibrium with transfers of $\Gamma|\sigma_{N\setminus D}$ for all nonempty $D \subsetneq N$. $\sigma \in \prod_{j\in N} S_j$ is a *coalition-proof Nash equilibrium with transfers of* Γ if it is self-enforcing with transfers in Γ and there are no other self-enforcing strategies with transfers $\sigma' \in \prod_{j\in N} S_j$ in Γ and $(\tau_j)_{j\in N} \in \mathbb{R}^n$ such that $\sum_{j\in N} \tau_j = 0$ and $V_i(\sigma) < V_i(\sigma'_D, \sigma_{N\setminus D}) + \tau_i$ for all $i \in N$.

Note that $\sigma \in \prod_{j\in N} S_j$ is a coalition-proof Nash equilibrium with transfers of Γ if and only if it is self-enforcing with transfers in Γ and there are no self-enforcing strategies with transfers $\sigma' \in \prod_{j\in N} S_j$ such that $\sum_{j\in N} V_j(\sigma) < \sum_{j\in N} V_j(\sigma')$.

Regarding the strong Nash equilibrium, since monetary transfers increase the possibility of coalition deviations, every strong Nash equilibrium with transfers is generally a strong Nash equilibrium, but the converse is not necessarily true. However, the same does not apply to a coalition-proof Nash equilibrium. The two sets of coalition-proof Nash equilibria may be disjointed. See Appendix C.

Remark 1 The remarks on the above equilibria are in order. (i) Every strong Nash equilibrium with transfers is a strong Nash equilibrium, which in turn is a coalition-proof Nash equilibrium. (ii) Every strong Nash equilibrium with transfers is a coalition-proof Nash equilibrium with transfers. (iii) In Γ, no coalition-proof Nash equilibrium is Pareto-dominated by other coalition-proof Nash equilibria. (iv) There are never two distinct coalition-proof Nash equilibria with transfers that take different values of the sum of the payoffs to agents.

3 Results: Nonharmful public projects

We consider an economy in which agents undertake a project that is *nonharmful* for all agents in the sense that the increase in project level does not harm any agent. This economy is formally defined as a list $[N, (u_i)_{i\in N}, c]$ in which u_j is *weakly increasing* in the project level for all $j \in N$: for all $j \in N$ and all $y \in \mathcal{Y}$,

$$\Delta u_j(y+1, y) \geq 0 \tag{3}$$

and c is weakly convex and increasing in level (see (1)). We refer to this economy as $\mathbf{e^1}$.

Theorem 1 For an economy $\mathbf{e^1} = [N, (u_i)_{i\in N}, c]$, in the unit-by-unit contribution game, (i) there is no Nash equilibrium at which the project is undertaken over level y^* and (ii) the set of Nash equilibria at which the project is undertaken at level y^* coincides with the sets of strong Nash equilibria with and without transfers and the sets of coalition-proof Nash equilibria with and without transfers, and all sets are nonempty.

The proof is provided in the appendix. The project levels at Nash equilibria may be multiple, but at most y^*.[9] Since strong Nash equilibria and coalition-proof Nash equilibria single out Nash equilibria with

[9] We can make an example in which the unit-by-unit contribution game may have Nash equilibria at which the project is undertaken below y^*. For example, consider a case of $\mathcal{Y} = \{0, 1, 2\}$, $c(y) = 10y$ for all $y \in \mathcal{Y}$, $N = \{1, 2\}$, and $u_i(1) = 7$ and

efficient project implementation levels, coordination possibilities modeled through those equilibria successfully lead to efficient allocation. In this sense, given coordination possibilities, the unit-by-unit contribution mechanism is successful in the implementation of nonharmful projects.

Studies on the provision of integer-unit public goods have examined several distinct benefit functions. Bagnoli and Lipmann (1989) and Nishimura and Shinohara (2013) assume that agents' benefit functions are strictly increasing in the public good level. Moreover, Bagnoli and Lipmann (1989) impose strict concavity on the benefit functions. Brânzei et al. (2005) and Shinohara (2014) assume that every agent $i \in N$ has a discontinuous benefit function such that there is a threshold level of the public good y_i and a positive constant value $\bar{u}_i$ such that $u_i(y) = \bar{u}_i$ if $y \geq y_i$ and $u_i(y) = 0$ otherwise. Obviously, all of the benefit functions in the literature are examples of weakly increasing benefit functions. The existence of Nash equilibria with efficient projects, shown by Bagnoli and Lipmann (1989) and Brânzei et al. (2005), is extensible to the case in which agents have weakly increasing benefit functions.

By Theorem 1, we observe that the Nash equilibrium with an efficient project is robust to several types of coalitional deviations. This robustness property is stronger than the finding by Brânzei et al. (2005). This is because while Brânzei et al. (2005) examine a strong Nash equilibrium (without transfers), we examine four refined Nash equilibria, including a strong Nash equilibrium.[10]

4 Results: Harmful public projects

To what extent are the desirable properties of the unit-by-unit contribution mechanism, shown in Theorem 1, satisfied when implementing a public project that is sometimes *harmful* to some agents? We consider an economy in which at least one agent has a benefit function that is not weakly increasing, that is, an economy $[N, (u_i)_{i \in N}, c]$ in which there exist $j \in N$ and $y_j \in \mathcal{Y} \backslash \{\bar{y}\}$ such that $\Delta u_j(y_j + 1, y_j) < 0$ and c satisfies (1). In this economy, some agents such as agent j above do not always benefit from an increase in the project level.

Firstly, we provide examples to show that in this economy, the unit-by-unit contribution mechanism may not achieve an efficient project level at some refined Nash equilibria.

Example 1 Let $\mathcal{Y} = \{0, 1, 2\}$. Let $c(y) = 10y$ for all $y \in \mathcal{Y}$. Let $N = \{1, 2\}$. Suppose $u_1(1) = 4$, $u_1(2) = 1$, $u_2(1) = 12$, and $u_2(2) = 23$. Then, $y^* = 1$ and $Y^{\max} = 2$. Firstly, we show that no Nash equilibrium supports the efficient undertaking of the project. Take $\sigma = (\sigma_1^1, \sigma_1^2; \sigma_2^1, \sigma_2^2)$ such that $\sigma_1^1 + \sigma_2^1 = 10$ and $\sigma_1^2 + \sigma_2^2 < 10$. In this σ, $y(\sigma) = 1$. However, it cannot be a Nash equilibrium because if agent 2 increases his marginal contribution to the second unit from σ_2^2 to $10 - \sigma_1^2$, then he is made better off (note that $\Delta u_2(2, 1) > \Delta c(2, 1) \geq \Delta c(2, 1) - \sigma_1^2$ in this example). We can easily verify that $\sigma' \in \prod_{j \in N} S_j$ such that $\sigma_1' = (0, 0)$ and $\sigma_2' = (10, 10)$ is a unique Nash equilibrium that is also coalition-proof. Secondly, we can verify that no strong Nash equilibrium exists since σ' is not a strong Nash equilibrium with or without transfers (consider a deviation by N from σ' to $\tilde{\sigma} \in \prod_{j \in N} S_j$ such that $(\tilde{\sigma}_1^1, \tilde{\sigma}_1^2) = (2, 0)$ and $(\tilde{\sigma}_2^1, \tilde{\sigma}_2^2) = (8, 0)$).

$u_2(2) = 13$ for all $i \in N$.

[10] In regard to the result in Brânzei et al. (2005), it would be important to discuss whether a Nash equilibrium with an efficient project achieves the core of some cooperative game. This is because Brânzei et al. (2005) show that utility allocations attained at strong Nash equilibria are the core of a cooperative game. We can show that if agents have weakly increasing benefit functions, all utility allocations at the strong Nash equilibria belong to the core of a cooperative game. The proof is available upon request.

Example 2 Let $\mathcal{Y} = \{0, 1, 2\}$ and $c(y) = 10y$. Let $N = \{1, 2, 3, 4\}$. Suppose that $u_1(1) = 7.5$ and $u_1(2) = 0$ and that $u_i(1) = 6$ and $u_i(2) = 12$ for all $i \in N\backslash\{1\}$. Then, $y^* = Y^{\max} = 2$. In this example, we show that there is no strong Nash equilibrium with transfers at which the project is undertaken at level y^*, while there exists a strong Nash equilibrium.

We can find a strategy profile that is a strong Nash equilibrium. For example, consider $\sigma \in \prod_{j \in N} S_j$ such that $(\sigma_1^1, \sigma_1^1) = (0, 0)$, $(\sigma_2^1, \sigma_2^1) = (10, 0)$, and $(\sigma_i^1, \sigma_i^1) = (0, 5)$ for $i = 3, 4$, which is a strong Nash equilibrium.

Secondly, we show that there exists no strong Nash equilibrium with transfers. Let σ be a Nash equilibrium such that $y(\sigma) = 2$. Since $u_1(2) = 0$ and σ is a Nash equilibrium, we obtain $(\sigma_1^1, \sigma_1^2) = (0, 0)$. We further obtain $\sum_{i \in N\backslash\{1\}} \sigma_i^y = \Delta c(y, y-1)$ for all $y \in \mathcal{Y}\backslash\{0\}$ (see Lemma A1 in Appendix A). At σ, $V_1(\sigma) = 0$ and $V_i(\sigma) = 12 - \sigma_i^1 - \sigma_i^2$ for all $i \in N\backslash\{1\}$. Now, we consider a coalition $\{1, j\}$ such that $j \in N\backslash\{1\}$ and $\sigma_j^2 > 0$. Suppose that this coalition deviates from σ to $\tilde{\sigma}_{\{1,j\}}$ such that $\tilde{\sigma}_1 = \sigma_1$ and $\tilde{\sigma}_j = (\sigma_j^1, 0)$. Then, $y(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}})$=1 and $V_1(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) + V_j(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) = 7.5 + 6 - \sigma_j^1$. Finally,

$$V_1(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) + V_j(\tilde{\sigma}_{\{1,j\}}, \sigma_{N\backslash\{1,j\}}) - \big(V_1(\sigma) + V_j(\sigma)\big) = 1.5 + \sigma_j^2 > 0.$$

Thus, no strong Nash equilibrium with transfers exists.

In these examples, there is only one agent whose benefit function is not weakly increasing. Nevertheless, the equilibria of the unit-by-unit contribution game have properties that are very different from those in Theorem 1. Firstly, a Nash equilibrium may not support the efficient project y^* (see Example 1). Secondly, strong Nash equilibria with and without transfers may not exist. Moreover, no strong Nash equilibrium with transfers may exist in either $Y^{\max} = y^*$ or $Y^{\max} > y^*$. Thirdly, although coalition-proof Nash equilibria with and without transfers exist in these examples, they do not always support an efficient project.

By Examples 1 and 2, the unit-by-unit contribution mechanism does not necessarily achieve an efficient project at refined Nash equilibria, unlike in the implementation of nonharmful projects. In particular, it is impossible for the mechanism to achieve efficiency through a strong Nash equilibrium, since it may not exist. We now focus on the coalition-proof Nash equilibria and examine to what extent the unit-by-unit contribution mechanism achieves an efficient project level in an economy with harmful projects.

The condition that at least one agent does not have a weakly increasing benefit function seems very weak, and hence we need to consider many economies for the analysis. For tractability, we focus on a subclass of such economies, in which agents have *weakly concave* benefit functions. Formally, we consider an economy $\mathbf{e}^2 = [N, (u_i)_{i \in N}, c]$ in which some agents do not have weakly increasing benefit functions; that is, there exist $j \in N$ and $y_j \in \mathcal{Y}\backslash\{\bar{y}\}$ such that

$$\Delta u_j(y_j + 1, y_j) < 0, \tag{4}$$

every agent has weakly concave benefit functions: for all $i \in N$ and all $y, y' \in \mathcal{Y}$ such that $\bar{y} > y > y'$,

$$\Delta u_i(y + 1, y) \leq \Delta u_i(y' + 1, y'), \tag{5}$$

and c is weakly convex and increasing (see (1)).

Note that there may be agents whose benefit functions are weakly increasing.

Lemma 2 In economy $\mathbf{e}^2$, for all $j \in N$, if u_j satisfies (4), then there exists $p(j) \in \mathcal{Y}\backslash\{\bar{y}\}$ such that

$$\begin{aligned} \Delta u_i(y+1, y) \geq 0 \text{ for all } y \in \mathcal{Y} \text{ such that } y < p(j) \\ \text{and } \Delta u_i(y+1, y) < 0 \text{ for all } y \in \mathcal{Y} \text{ such that } y \geq p(j). \end{aligned} \tag{6}$$

Proof. Suppose that u_j satisfies (4). By a weak concavity of u_j, $\Delta u_j(y+1, y) < 0$ for all $y \in \mathcal{Y}\backslash\{\bar{y}\}$ such that $y \geq y_j$. Thus, if there exists $y' \in \mathcal{Y}\backslash\{0, \bar{y}\}$ such that $\Delta u_j(y', y'-1) \geq 0$ and $\Delta u_j(y'+1, y') < 0$, then we can define $y' \equiv p(j)$. Otherwise, $p(j) \equiv 0$. ■

Our interpretation is that $p(j)$ is the level of the public project that peaks agent j's benefit from the project. In this economy, every agent whose benefit function is not weakly increasing has a *peaked benefit function*. For convenience, we also define this peak level of the project for all agents whose benefit function is weakly increasing as follows: $p(i) \equiv \bar{y}$ for all $i \in N$ such that $\Delta u_i(y+1, y) \geq 0$ for all $y \in \mathcal{Y}\backslash\{\bar{y}\}$. We further introduce some notations for the analysis. For all $y \in \mathcal{Y}$, let $N^y \equiv \{i \in N | \, p(i) \geq y\}$: the set of agents whose peak level is not less than y. Then, for all $i \in N$ and $y \in \mathcal{Y}\backslash\{0\}$, $i \in N^y$ if and only if $\Delta u_i(y, y-1) \geq 0$.

In this economy, although y^* is a unique efficient level of the project, $Y^{\max}$ may not be equal to y^*. We provide a necessary and sufficient condition under which $Y^{\max} = y^*$ in Lemma 3, which would be useful for subsequent analyses.

Lemma 3 In economy $\mathbf{e}^2$, $Y^{\max} = y^*$ if and only if

$$\sum_{j \in N^{y^*+1}} \Delta u_j(y^*+1, y^*) < \Delta c(y^*+1, y^*). \tag{7}$$

Proof. ($\Leftarrow$) By the definition of $Y^{\max}$, $y^* \leq Y^{\max}$. Suppose, to the contrary, that $y^*+1 \leq Y^{\max}$. Let $D \subseteq N$ be such that $Y(D) = Y^{\max}$. Then, $\sum_{j \in D} \Delta u_j(Y(D), y^*) > \Delta c(Y(D), y^*)$.

Note that $\sum_{j \in D} \Delta u_j(y^*+1, y^*) \leq \sum_{j \in N^{y^*+1}} \Delta u_j(y^*+1, y^*)$. This inequality, together with (1), (5), and (7), implies that for all $y \in \mathcal{Y}$ such that $y \geq y^*+1$,

$$\sum_{j \in D} \Delta u_j(y+1, y) \leq \sum_{j \in D} \Delta u_j(y^*+1, y^*) < \Delta c(y^*+1, y^*) \leq \Delta c(y+1, y).$$

Finally, we obtain $\sum_{j \in D} \Delta u_j(Y(D), y^*) < \Delta c(Y(D), y^*)$, which is a contradiction.

($\Rightarrow$) $Y^{\max} = y^*$ implies that $Y(N^{y^*+1}) \leq y^*$. Since $Y(N^{y^*+1})$ is the unique maximizer of $\sum_{j \in N^{y^*+1}} u_j(y) - c(y)$,

$$\sum_{j \in N^{y^*+1}} \Delta u_j(Y(N^{y^*+1})+1, Y(N^{y^*+1})) < \Delta c(Y(N^{y^*+1})+1, Y(N^{y^*+1})).$$

By (1), (5), and $Y(N^{y^*+1}) \leq y^*$,

$$\sum_{j \in N^{y^*+1}} \Delta u_j(y^*+1, y^*) \leq \sum_{j \in N^{y^*+1}} \Delta u_j(Y(N^{y^*+1})+1, Y(N^{y^*+1}))$$

and

$$\Delta c(Y(N^{y^*+1})+1, Y(N^{y^*+1})) \leq \Delta c(y^*+1, y^*).$$

Thus, we obtain (7). ■

Lemma 4 is a preparation for subsequent analyses.

Lemma 4 In economy $\mathbf{e}^2$, $\sum_{i \in N^{y^*}} \Delta u_i(y, y-1) > \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y^*$.

Proof. By Lemma 1, $\sum_{i \in N} \Delta u_i(y^*, y^*-1) > \Delta c(y^*, y^*-1)$. Since $\sum_{i \notin N^{y^*}} \Delta u_i(y^*, y^*-1) < 0$ (if $N^{y^*} \neq \emptyset$),

$$\sum_{i \in N} \Delta u_i(y^*, y^*-1) = \sum_{i \in N^{y^*}} \Delta u_i(y^*, y^*-1) + \sum_{i \notin N^{y^*}} \Delta u_i(y^*, y^*-1) \leq \sum_{i \in N^{y^*}} \Delta u_i(y^*, y^*-1).$$

Thus, $\sum_{i \in N^{y^*}} \Delta u_i(y^*, y^*-1) > \Delta c(y^*, y^*-1)$. By (1) and (5), we obtain the condition in the statement of this lemma. ■

4.1 Nash equilibria and Pareto efficiency

As Examples 1 and 2 show, the unit-by-unit contribution game may or may not have a Nash equilibrium that undertakes the project efficiently in economy $\mathbf{e}^2$. Hence, we investigate under which conditions the unit-by-unit contribution game has such a Nash equilibrium in $\mathbf{e}^2$.

By Lemma 4, we can construct a strategy profile $\sigma^* \in \prod_{j \in N} S_j$ such that:

$$\begin{aligned} &\text{If } y^*+1 \leq y \leq \bar{y}, \text{ then } \sigma_i^{*y} = 0 \text{ for all } i \in N. \\ &\text{If } 1 \leq y \leq y^*, \text{then } \sigma_i^{*y} = 0 \text{ for all } i \notin N^{y^*}, 0 \leq \sigma_i^{*y} < \Delta u_i(y, y-1) \text{ for all } i \in N^{y^*}, \\ &\text{and } \sum_{i \in N^{y^*}} \sigma_i^{*y} = \Delta c(y, y-1). \end{aligned} \tag{8}$$

At σ^*, the project is undertaken efficiently. However, as we can easily check, while it is a Nash equilibrium in Example 2, it is not in Example 1. Hence, some conditions are needed for it to be a Nash equilibrium. Profile σ^* plays an important role in establishing a necessary and sufficient condition for a Nash equilibrium to achieve an efficient project level.

Proposition 1 *In economy* $\mathbf{e}^2$*, the unit-by-unit contribution game has a Nash equilibrium at which the project is undertaken efficiently if and only if*

$$\Delta u_i(y^*+1, y^*) \leq \Delta c(y^*+1, y^*) \textit{ for all } i \in N. \tag{9}$$

The proof is provided in the appendix. Whether a Nash equilibrium with an efficient project level exists depends on the relationship between the marginal benefits of each agent and marginal cost for the y^*+1-th unit of the project. We can intuitively understand (9). If (9) holds, no agent gains from a one-unit increase in the project level from y^* units. Hence, without (9), no Nash equilibrium supports an efficient project level.

Corollary 1 establishes a sufficient condition under which the unit-by-unit contribution game has a Nash equilibrium with an efficient project.

Corollary 1 In economy $\mathbf{e}^2$, if $Y^{\max} = y^*$, then there is a Nash equilibrium at which the project is undertaken at the level y^*.

Proof. $Y^{\max} = y^*$ implies (7), which in turn implies (9). Thus, by Proposition 1, a Nash equilibrium exists such that the public project is provided efficiently. ■

It is easily seen that $Y^{\max} = y^*$ is not a necessary condition for a Nash equilibrium to support an efficient project.

As Example 1 shows, in economy $\mathbf{e}^2$, there may be a Nash equilibrium that supports over-implementation of the project in the unit-by-unit contribution game. Proposition 2 provides a necessary and sufficient condition for an economy under which the project is implemented over the efficient level at a Nash equilibrium.

Proposition 2 *In economy* $\mathbf{e}^2$*, there exists a Nash equilibrium at which the project is undertaken over the efficient level* y^* *if and only if* $Y^{\max} > y^*$.

The proof is provided in the appendix. We can intuitively interpret $Y^{\max} > y^*$. $Y^{\max} > y^*$ if and only if $Y(D) > y^*$ for some $D \subsetneq N$. Coalition D can undertake the public project at the level $Y(D)$ by itself since $\sum_{j \in D} \Delta u_j(y, y-1) \geq \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y(D)$. Thus, there exists a proper subgroup of N that can undertake the project over y^*. In conclusion, whether over-implementation of the project is supported at a Nash equilibrium depends on the existence of such a subgroup.

Theorem 2 summarizes the results in subsection 4.1, which is derived directly from Corollary 1 and Proposition 2.

Theorem 2 *In economy* $\mathbf{e}^2$*, (i) if* $Y^{\max} = y^*$*, then there exists a Nash equilibrium that supports the efficient project, but there is no Nash equilibrium at which the project is undertaken over the efficient level. (ii) If* $Y^{\max} > y^*$*, there exists a Nash equilibrium with over-implementation of the project, and there may be a Nash equilibrium that supports an efficient project.*

Remarks about Theorem 2 are in order. Firstly, in the case of $Y^{\max} = y^*$, there may be a Nash equilibrium that supports underprovision of the public project. Secondly, in the case of $Y^{\max} > y^*$, there may be a Nash equilibrium with efficient project implementation. Thus, as in economy $\mathbf{e}^1$, the unit-by-unit contribution mechanism may face a multiplicity of Nash equilibria that support different project levels in economy $\mathbf{e}^2$.

4.2 Coalition-proof Nash equilibria and Pareto efficiency

We examine which Nash equilibria are coalition-proof both with and without transfers in the unit-by-unit contribution game in economy $\mathbf{e}^2$. Lemma 5 is a preliminary for the analysis.

Lemma 5 There exists $\tilde{M} \subseteq N$ that satisfies

$$\Delta u_j(y, y-1) > 0 \text{ for all } j \in \tilde{M} \text{ and for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}, \tag{10}$$

$$\text{and} \quad \sum_{j \in \tilde{M}} \Delta u_j(y, y-1) > \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}. \tag{11}$$

The proof is provided in the appendix. Let $\mathcal{M} \subseteq N$ be the "largest" set that satisfies (10) and (11) in the sense that no other sets satisfy these conditions or include $\mathcal{M}$. By these conditions, we can find

$\sigma \in \mathbb{R}_+^{n\bar{y}}$ such that

$$\begin{aligned}&\text{If } Y^{\max}+1 \le y \le \bar{y}, \text{ then } \sigma_i^y = 0 \text{ for all } i \in N.\\ &\text{If } 1 \le y \le Y^{\max}, \text{ then } \sigma_i^y = 0 \text{ for all } i \notin \mathcal{M},\ 0 < \sigma_i^y < \Delta u_i(y, y-1) \text{ for all } i \in \mathcal{M},\\ &\text{and } \sum_{j \in \mathcal{M}} \sigma_j^y = \Delta c(y, y-1).\end{aligned} \tag{12}$$

By (12), for all $j \in \mathcal{M}$,

$$\Delta u_j(Y^{\max}, \hat{y} - 1) > \sum_{y=\hat{y}}^{Y^{\max}} \sigma_j^y \text{ for all } \hat{y} \in \mathcal{Y} \text{ such that } 1 \le \hat{y} \le Y^{\max}. \tag{13}$$

In the proof of Proposition 3, provided in the appendix, we show that σ in (12) is a coalition-proof Nash equilibrium with and without transfers.

Proposition 3 *In economy $\mathbf{e}^2$, the unit-by-unit contribution game has coalition-proof Nash equilibria with and without transfers at which the project is undertaken at the level $Y^{\max}$.*

The proof is provided in the appendix. Although the unit-by-unit contribution game may not have strong Nash equilibria with or without transfers, it always has coalition-proof Nash equilibria with and without transfers. The self-enforcing property of coalition deviations guarantees the existence of coalition-proof Nash equilibria. It would be useful to again consider Example 1 in order to intuitively understand how this property works. Recall that $y^* = 1$ and $Y^{\max} = 2$ in this example. Recall also that $\sigma' = (\sigma_1'^1, \sigma_1'^2; \sigma_2'^1, \sigma_2'^2) = (0, 0; 10, 10)$ is the Nash equilibrium, but it is not a strong Nash equilibrium because N has a profitable deviation $\tilde{\sigma} = (\tilde{\sigma}_1^1, \tilde{\sigma}_1^2; \tilde{\sigma}_2^1, \tilde{\sigma}_2^2) = (2, 0; 8, 0)$ from σ'. By this deviation, the project level declines from $y(\sigma') = 2$ to $y(\tilde{\sigma}) = 1$. However, this deviation is not self-enforcing because agent 2 is willing to get the project level back to two units after the deviation. This is because agent 2's marginal benefit from the second unit is greater than the marginal cost for the unit. When the self-enforcing property matters, agents 1 and 2 do not agree with the first joint deviation to decrease the project level. In general, in the unit-by-unit contribution game in economy $\mathbf{e}^2$, the deviation to decrease the project level from $Y^{\max}$ is not self-enforcing (see the proof of Proposition 3 for details).

The next proposition shows that no coalition-proof Nash equilibrium supports the public project under $Y^{\max}$.

Proposition 4 *Suppose that there exists a Nash equilibrium $\hat{\sigma} \in \mathbb{R}_+^{n\bar{y}}$ such that $y(\hat{\sigma}) < Y^{\max}$ in the unit-by-unit contribution game in economy $\mathbf{e}^2$. Then, $\hat{\sigma}$ is not a coalition-proof Nash equilibrium with or without transfers.*

The proof is provided in the appendix. As Proposition 3 shows, there is a coalition-proof Nash equilibrium at which the project is undertaken at the level $Y^{\max}$. This, together with (13), implies that agents in $\mathcal{M}$ have a self-enforcing deviation from $\hat{\sigma}$ in a way that increases the project level from $y(\hat{\sigma})$ to $Y^{\max}$ and makes all of them better off (see the proof of Proposition 4 for details).

Theorem 3 summarizes under what condition a coalition-proof Nash equilibrium achieves the efficient project level.

Theorem 3 *In economy* $\mathbf{e}^2$*, (i) the project is undertaken at the level* y^* *at all coalition-proof Nash equilibria with and without transfers if* $Y^{\max} = y^*$*. (ii) The project is undertaken over the efficient level at all coalition-proof Nash equilibria if* $Y^{\max} > y^*$*.*

The proof is provided in the appendix. When the project is harmful for some agents, a Nash equilibrium itself may not support an efficient project. In some cases, multiple Nash equilibria support both efficient implementation and over-implementation of the project. In these cases, no Nash equilibrium with an efficient project is robust to coalition deviations. No strong Nash equilibria may exist. Coalition-proof Nash equilibria with and without transfers always exist, but they single out the Nash equilibrium with over-implementation of the project. These results differ greatly from those for non-harmful project implementation.

4.3 A modified mechanism

In the discussion after Proposition 2, we mention that $Y^{\max} > y^*$ means the existence of a group that can over-implement the project. If such a group exists, agents outsider this group cannot prevent the over-implementation of the project because they can announce only nonnegative contributions for each one-unit increase. Now, we modify the unit-by-unit contribution mechanism in such a way that agents can announce negative numbers for each one-unit increase. Let $S_i \equiv \mathbb{R}^{\bar{y}}$ for each $i \in N$. For each $\sigma \in \prod_{j \in N} S_j$, the level of the project $y(\sigma)$ is defined in the same way as (2). For each $i \in N$ and $(\sigma_i^y)_{y \in \mathcal{Y} \setminus \{0\}} \in S_i$, $(t_i^y(\sigma_i^y))_{y \in \mathcal{Y} \setminus \{0\}}$ is defined as a vector of i's *actual* contributions for each one-unit increase such that for each $y \in Y \setminus \{0\}$, $t_i^y(\sigma_i^y) = \sigma_i^y$ if $\sigma_i^y \geq 0$ and $t_i^y(\sigma_i^y)$ takes a positive value otherwise. For each $\sigma \in \prod_{j \in N} S_j$ and each $i \in N$, $V_i(\sigma) = u_i(y(\sigma)) - \sum_{y \in Y \setminus \{0\}} t_i^y(\sigma_i^y)$. Note that by this modification, agents can prevent any one-unit increase if they announce sufficiently small negative numbers.

We can intuitively understand this modified mechanism in which for each one-unit increase, each agent is asked to announce a "willingness to pay" for carrying out the increase (a positive number) or that for preventing it (a negative number). Whether the project increases by one unit depends upon the sum of the announced willingness-to-pay. For any one-unit increase, if agents announce a positive value for that increase, they make the same payment as their announcement. Otherwise, their actual payment for that increase can be any positive value. There are some examples concerning how to set the actual contributions for negative numbers. For example, consider that $t_i^y(\sigma_i^y) = |\sigma_i^y|$ when $\sigma_i^y < 0$. In this example, agents who announce a negative number for some one-unit increase pay the absolute value of their willingness to pay for preventing that increase. We can consider another example in which for each $y \in \mathcal{Y} \setminus \{0\}$, $t_i^y(\sigma_i^y) = \varepsilon$ for some positive constant ε when $\sigma_i^y < 0$. We can intuitively understand that ε is a "fine" for announcing a negative number. It is enough that ε is very close to zero.

Proposition 5 *In some economy* $\mathbf{e}^2$*, no Nash equilibrium supports the efficient implementation of the public project in the modified unit-by-unit contribution mechanism.*

The proof is provided in the appendix.

In this mechanism, if agents announce negative values for some one-unit increase, then their payment for that increase can take any positive value. In this sense, this modified mechanism seems to constitute a reasonably large class of modifications of the unit-by-unit contribution mechanisms. By

Proposition 5, we confirm that introducing negative contributions to the unit-by-unit contribution mechanism is not sufficient for the efficient undertaking of the project. We also confirm that if agents announce negative numbers for some one-unit increase, then they are subsidized to some extent, but not asked to contribute.

5 Conclusion

The unit-by-unit contribution mechanism seems suitable for the implementation of integer-unit public projects and applicable, to some extent, to public project initiatives in the real world. Hence, it is important that we understand how this mechanism works in the implementation of various public projects. However, this issue has received only limited attention. Our aim is to examine to what extent this mechanism achieves Pareto efficiency in the implementation of public projects. We consider not only a project that is nonharmful for all agents but also one that is not.

Our results are as follows. The mechanism works well in an economy in which the project is nonharmful for all agents. In this economy, the mechanism achieves an efficient project level only at a strong Nash equilibrium and a coalition-proof Nash equilibrium with and without transfers. In this sense, given various coalitional behaviors, the mechanism achieves efficiency. On the other hand, in other economies, the mechanism does not always work well. When the project is harmful for some agents, the unit-by-unit contribution mechanism does not necessarily have a Nash equilibrium with an efficient project. Even if the mechanism has such a Nash equilibrium, it is not necessarily supported at a strong Nash equilibrium or a coalition-proof Nash equilibrium. We introduce a reasonable class of modified unit-by-unit contribution mechanisms, but no mechanism in this class achieves an efficient public project in Nash equilibria. We conclude that the unit-by-unit contribution mechanism should be used only for public projects that benefit all agents. In order to achieve an efficient project level that is harmful for some, we need to consider another class of modified unit-by-unit contribution mechanisms or construct a completely new mechanism to undertake public projects. This is left for future research.

Appendix A: Preliminary results

In Appendix A, we examine a unit-by-unit contribution game without (1), (3), or (5). Instead of these conditions, we impose other conditions on the benefit and cost functions in each of subsequent lemmas. The results obtained in this appendix are applied to prove the results in the main text.

Let $\Gamma^0 = [\mathcal{N}, (S_i, \mathcal{V}_i)_{i\in\mathcal{N}}]$ be a unit-by-unit contribution game where $\mathcal{N}$ is the set of agents, S_i is i's set of strategies such that $S_i = \mathbb{R}_+^{\bar{y}}$, and $\mathcal{V}_i : \prod_{j\in\mathcal{N}} S_j \to \mathbb{R}$ is i's payoff function such that $\sigma_{\mathcal{N}} \in \prod_{j\in\mathcal{N}} S_j \longmapsto \mathcal{V}_i(\sigma_{\mathcal{N}}) \equiv u_i(y(\sigma_{\mathcal{N}})) - \sum_{y=1}^{\bar{y}} \sigma_i^y \in \mathbb{R}$ where $y : \prod_{j\in\mathcal{N}} S_j \to \mathcal{Y}$ is a mapping assigning a level of the public project to each strategy profile, which is defined in the same way as (2) in the main text. We assume that for all $y \in \mathcal{Y}$, $c(y) \geq 0$ and for all $y \in \mathcal{Y}\backslash\{0\}$, $\Delta c(y, y-1) \equiv \max\{0, c(y)-c(y-1)\}$. However, we do not impose any of (1), (3), and (5) on Γ^0.

A.1 Results of Nash equilibria of Γ^0

Lemma A1 shows that the contributions at every Nash equilibrium satisfy the budget balance condition.

Lemma A1 Suppose that $\sigma_{\mathcal{N}} \in \mathbb{R}_{+}^{|\mathcal{N}|\bar{y}}$ is a Nash equilibrium of Γ^0. Then,

$$\sum_{j \in \mathcal{N}} \sigma_j^y = \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y(\sigma_{\mathcal{N}}) \text{ if } y(\sigma_{\mathcal{N}}) \geq 1. \tag{14}$$

$$\sigma_j^y = 0 \text{ for all } j \in \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq y(\sigma_{\mathcal{N}}) + 1 \text{ if } y(\sigma_{\mathcal{N}}) + 1 \leq \bar{y}. \tag{15}$$

Proof. *Proof of* (14). Since the project is undertaken at the level $y(\sigma_{\mathcal{N}})$ at $\sigma_{\mathcal{N}}$, $\sum_{j \in \mathcal{N}} \sigma_j^y \geq \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y(\sigma)$. Suppose, to the contrary, that there exists $\tilde{y} \in \mathcal{Y}$ such that $1 \leq \tilde{y} \leq y(\sigma)$ and $\sum_{j \in \mathcal{N}} \sigma_j^{\tilde{y}} > \Delta c(\tilde{y}, \tilde{y}-1)$. Then, clearly, there exists $i \in \mathcal{N}$ such that $\sigma_i^{\tilde{y}} > 0$. Even if this agent i decreases his contribution to $\tilde{y}$-th unit from $\sigma_i^{\tilde{y}}$ to $\sigma_i'^{\tilde{y}} = \max\left\{0,\ \Delta c(\tilde{y}, \tilde{y}-1) - \sum_{j \in \mathcal{N} \setminus \{i\}} \sigma_j^{\tilde{y}}\right\}$, he can still enjoy the project at the level $y(\tilde{\sigma}_{\mathcal{N}})$ while his total contribution decreases. Hence, he is made better off by this deviation, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium.

Proof of (15). Suppose that there exist $j \in N$ and $\tilde{y} \in \mathcal{Y}$ such that $\tilde{y} \geq y(\sigma_{\mathcal{N}})+1$ and $\sigma_j^{\tilde{y}} > 0$. If agent j switches from $\sigma_j^{\tilde{y}}$ to $\sigma_j'^{\tilde{y}} = 0$, the level of the project does not change. Hence, by this switch, agent j can still enjoy the project at the level $y(\sigma_{\mathcal{N}})$ as well as reduce his contribution, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium. ■

Lemma A2 proves that at every Nash equilibrium, under some condition, marginal contributions do not exceed the marginal benefit from the increase of the public project.

Lemma A2 Suppose that $\sigma_{\mathcal{N}} \in \mathbb{R}_{+}^{|\mathcal{N}|\bar{y}}$ is a Nash equilibrium of Γ^0. Suppose also that $\Delta u_j(y, y-1) \geq 0$ for all $j \in \mathcal{N}$ and all $y \in \mathcal{Y}$ such that $1 \leq y \leq y(\sigma_{\mathcal{N}})$. Then,

$$\Delta u_j(y(\sigma_{\mathcal{N}}), y') \geq \sum_{y=y'+1}^{y(\sigma)} \sigma_j^y \text{ for all } j \in \mathcal{N} \text{ and for all } y' \in \mathcal{Y} \text{ such that } y' \leq y(\sigma_{\mathcal{N}}) - 1. \tag{16}$$

Proof. The proof is obtained by induction. Let $i \in N$. Suppose, to the contrary, that

$$\Delta u_i\left(y(\sigma_{\mathcal{N}}), y(\sigma_{\mathcal{N}}) - 1\right) < \sigma_i^{y(\sigma_{\mathcal{N}})}.$$

Then, $\sigma_i^{y(\sigma_{\mathcal{N}})} > 0$. If i reduces his contribution to the $y(\sigma)$-th unit to zero, the level of the project decreases to $y(\sigma_{\mathcal{N}})-1$ by (14) of Lemma A1 and his payoff increases by $\sigma_i^{y(\sigma_{\mathcal{N}})} - \Delta u_i\left(y(\sigma_{\mathcal{N}}), y(\sigma_{\mathcal{N}}) - 1\right) > 0$, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium.

Let $y' \in \mathcal{Y}$ be such that $1 \leq y' \leq y(\sigma_{\mathcal{N}})-1$. Suppose, as an induction hypothesis, that $\Delta u_i(y(\sigma_{\mathcal{N}}), y') \geq \sum_{y=y'+1}^{y(\sigma_{\mathcal{N}})} \sigma_i^y$. Then, we show that $\Delta u_i\left(y(\sigma_{\mathcal{N}}), y'-1\right) \geq \sum_{y=y'}^{y(\sigma_{\mathcal{N}})} \sigma_i^y$. Suppose, to the contrary, that $\Delta u_i(y(\sigma_{\mathcal{N}}), y'-1) < \sum_{y=y'}^{y(\sigma_{\mathcal{N}})} \sigma_i^y$. By this inequality,

$$\Delta u_i\left(y(\sigma_{\mathcal{N}}), y'\right) - \sum_{y=y'+1}^{y(\sigma_{\mathcal{N}})} \sigma_i^y + \Delta u_i(y', y'-1) < \sigma_i^{y'}.$$

By this condition and the induction hypothesis, $\Delta u_i(y', y'-1) < \sigma_i^{y'}$. By $\Delta u_i(y', y'-1) \geq 0$, we obtain $\sigma_i^{y'} > 0$. Let $\tilde{\sigma}_i \in \mathbb{R}_{+}^{\bar{y}}$ be i's deviation strategy from σ such that $\tilde{\sigma}_i^y = \sigma_i^y$ if $1 \leq y \leq y'-1$ and $\tilde{\sigma}_i^y = 0$ otherwise. Since $\sigma_i^{y'} > 0$, the project is undertaken at the level $y'-1$ at $(\tilde{\sigma}_i, \sigma_{\mathcal{N}\setminus\{i\}})$ and $\mathcal{V}_i(\tilde{\sigma}_i, \sigma_{\mathcal{N}\setminus\{i\}}) = u_i(y'-1) - \sum_{y=1}^{y'-1} \sigma_i^y$. We obtain $\mathcal{V}_i(\sigma_{\mathcal{N}}) - \mathcal{V}_i(\tilde{\sigma}_i, \sigma_{\mathcal{N}\setminus\{i\}}) = \Delta u_i\left(y(\sigma_{\mathcal{N}}), y'-1\right) - \sum_{y=y'}^{y(\sigma_{\mathcal{N}})} \sigma_i^y < 0$, which contradicts the supposition that $\sigma_{\mathcal{N}}$ is a Nash equilibrium. ■

Lemma A3 provides a sufficient condition of a Nash equilibrium in Γ^0.

Lemma A3 Let $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$. Suppose that for all $j \in \mathcal{N}$,

$$\Delta u_j(y, y-1) \geq 0 \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y(\sigma_{\mathcal{N}}) \tag{17}$$

$$\text{and} \quad \Delta u_j(y, y(\sigma_{\mathcal{N}})) \leq \Delta c(y, y(\sigma_{\mathcal{N}})) \text{ for all } y \in \mathcal{Y} \text{ such that } y \geq y(\sigma_{\mathcal{N}}) + 1. \tag{18}$$

Suppose also that $\sigma_{\mathcal{N}}$ satisfies (14)–(16). Then, $\sigma_{\mathcal{N}}$ is a Nash equilibrium of Γ^0.

Proof. Let $j \in \mathcal{N}$ and let $\hat{\sigma}_j \in \mathbb{R}_+^{\bar{y}}$ be a deviation strategy of j from σ. Firstly, we consider the case of $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \geq y(\sigma) + 1$. Since by this deviation, the level of the project increases from $y(\sigma)$ to $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$ and $\sigma_{\mathcal{N}}$ satisfies (15), then j contributes at least $\Delta c\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma)\right)$ to this increase. Moreover, by (14), j cannot reduce his contributions from the first to $y(\sigma)$-th unit to undertake the project at the level $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$. Thus,

$$\sum_{y=1}^{\bar{y}} \hat{\sigma}_j^y \geq \sum_{y=1}^{y(\sigma)} \sigma_j^y + \Delta c\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma)\right). \tag{19}$$

By (18), $\Delta c\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma)\right) \geq \Delta u_j\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma)\right)$. Then, by (19),

$$\sum_{y=1}^{\bar{y}} \hat{\sigma}_j^y \geq \sum_{y=1}^{y(\sigma)} \sigma_j^y + \Delta u_j\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}), y(\sigma)\right)$$

or, equivalently,

$$u_j(y(\sigma)) - \sum_{y=1}^{y(\sigma)} \sigma_j^y \geq u_j\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})\right) - \sum_{y=1}^{\bar{y}} \hat{\sigma}_j^y.$$

In this condition, the left-hand side is the payoff to j before the deviation and the right-hand side is the one after the deviation. Hence, j is not made better off by this deviation.

Secondly, we consider the case of $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \leq y(\sigma) - 1$. Since $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$ units of the project are undertaken,

$$\sigma_j^y \leq \hat{\sigma}_j^y \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) \text{ and } \sigma_j^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})+1} > \hat{\sigma}_j^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})+1}.$$

The maximal payoff to agent j by this deviation is $u_j\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})\right) - \sum_{y=1}^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})} \sigma_j^y$, which is obtained if $\hat{\sigma}_j^y = \sigma_j^y$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})$ and $\hat{\sigma}_j^y = 0$ for all $y \in \mathcal{Y}$ such that $y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}}) + 1 \leq y \leq \bar{y}$. The payoff to agent j before this deviation is $u_j(y(\sigma_{\mathcal{N}})) - \sum_{y=1}^{y(\sigma_{\mathcal{N}})} \sigma_j^y$, while that after the deviation is at most $u_j\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})\right) - \sum_{y=1}^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})} \sigma_j^y$. Clearly,

$$u_j(y(\sigma_{\mathcal{N}})) - \sum_{y=1}^{y(\sigma_{\mathcal{N}})} \sigma_j^y \geq u_j\left(y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})\right) - \sum_{y=1}^{y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})} \sigma_j^y$$

because $\Delta u_j\left(y(\sigma_{\mathcal{N}}), y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})\right) \geq \sum_{y=y(\hat{\sigma}_j, \sigma_{\mathcal{N}\setminus\{j\}})+1}^{y(\sigma_{\mathcal{N}})} \sigma_j^y$ by (16). Thus, agent j is not made better off by this deviation. ■

A.2 Results of strong Nash equilibria of Γ^0

Similarly to the main text, let $Y(\mathcal{D}) \in \arg\max_{y\in\mathcal{Y}} \sum_{j\in\mathcal{D}} u_j(y) - c(y)$ for all $\mathcal{D} \subseteq \mathcal{N}$ and let $Y^{\max} \equiv \max_{\mathcal{D}\subseteq\mathcal{N}} Y(\mathcal{D})$.

Lemma A4 Let $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$ be a Nash equilibrium such that $y(\sigma_{\mathcal{N}}) = Y^{\max}$. Suppose that

$$\sum_{j\in\mathcal{E}} \Delta u_j(y, Y^{\max}) \leq \Delta c(y, Y^{\max}) \text{ for all } \mathcal{E} \subseteq \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq Y^{\max} + 1. \tag{20}$$

If a coalition $\mathcal{D} \subseteq \mathcal{N}$ has deviation strategies $\sigma'_{\mathcal{D}} \in \mathbb{R}_+^{|\mathcal{D}|\bar{y}}$ such that $y(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}}) \geq Y^{\max} + 1$, then $\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}}) \geq \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$.

Proof. Let $y' \equiv y(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$. Suppose, to the contrary, that $\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}}) > \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}})$. By this inequality,

$$\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) > \sum_{j\in\mathcal{D}} \sum_{y=1}^{Y^{\max}} (\sigma'^y_j - \sigma^y_j) + \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma'^y_j.$$

Since the project is undertaken at the level y' and $Y^{\max} < y'$, we obtain $\sum_{j\in\mathcal{D}} \sum_{y=1}^{Y^{\max}} (\sigma'^y_j - \sigma^y_j) \geq 0$; otherwise, y' units are never provided. Consequently,

$$\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) > \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma'^y_j, \tag{21}$$

On the other hand, by (20), we obtain $\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) \leq \Delta c(y', Y^{\max})$. Since the project is undertaken at the level y' by this deviation and $y' \leq \bar{y}$,

$$\Delta c(y', Y^{\max}) \leq \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{y'} \sigma'^y_i \leq \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma'^y_i.$$

Thus,

$$\sum_{j\in\mathcal{D}} \Delta u_j(y', Y^{\max}) \leq \sum_{j\in\mathcal{D}} \sum_{y=Y^{\max}+1}^{\bar{y}} \sigma'^y_i,$$

which contradicts (21). ■

Lemma A5 Suppose that (20) and

$$\Delta u_j(y, y-1) \geq 0 \text{ for all } j \in \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}. \tag{22}$$

Then, every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers of Γ^0.

Proof. Let $\sigma_{\mathcal{N}} \in \mathbb{R}_+^{|\mathcal{N}|\bar{y}}$ be a Nash equilibrium such that $y(\sigma_{\mathcal{N}}) = Y^{\max}$. By (22) and Lemmas A1 and

A2,

$$\sum_{j\in\mathcal{N}} \sigma_j^y = \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max},$$

$$\sigma_j^y = 0 \text{ for all } j \in \mathcal{N} \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq y(Y^{\max}) + 1, \text{ and} \tag{23}$$

$$\Delta u_j(Y^{\max}, \hat{y}) \geq \sum_{y=\hat{y}+1}^{Y^{\max}} \sigma_j^y \text{ for all } j \in \mathcal{N} \text{ and for all } \hat{y} \in \mathcal{Y} \text{ such that } \hat{y} \leq Y^{\max} - 1. \tag{24}$$

Suppose, to the contrary, that there exists a coalition $\mathcal{D} \subseteq \mathcal{N}$ and $\sigma'_{\mathcal{D}} \in \mathbb{R}_+^{|\mathcal{D}|\bar{y}}$ such that

$$\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}}) < \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}}). \tag{25}$$

Let $y' \in \mathcal{Y}$ be the level of the public project that deviates by $\mathcal{D}$: $y' \equiv y(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$. By Lemma A4, if $y' \geq Y^{\max} + 1$, then it is impossible that $\sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma_{\mathcal{N}}) < \sum_{j\in\mathcal{D}} \mathcal{V}_j(\sigma'_{\mathcal{D}}, \sigma_{\mathcal{N}\setminus\mathcal{D}})$. It is trivial that if $y' = Y^{\max}$, then the deviation by $\mathcal{D}$ is not improving. Finally, we need to consider the case of $y' \leq Y^{\max} - 1$. By (25),

$$\sum_{j\in\mathcal{D}} \Delta u_j(Y^{\max}, y') < \sum_{j\in\mathcal{D}} \sum_{y=1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y).$$

Since the deviation by $\mathcal{D}$ attains y', $\sum_{j\in\mathcal{D}} \sum_{y=1}^{y'} \sigma_j^y \leq \sum_{j\in\mathcal{D}} \sum_{y=1}^{y'} \sigma_j'^y$. By this inequality,

$$\begin{aligned} \sum_{j\in\mathcal{D}} \sum_{y=1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y) &= \sum_{j\in\mathcal{D}} \sum_{y=1}^{y'} (\sigma_j^y - \sigma_j'^y) + \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y) \\ &\leq \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{\bar{y}} (\sigma_j^y - \sigma_j'^y) \leq \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{\bar{y}} \sigma_j^y = \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y. \end{aligned}$$

The last equality follows from (23). Combining these two conditions yields

$$\sum_{j\in\mathcal{D}} \Delta u_j(Y^{\max}, y') < \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y.$$

However, by (24), $\sum_{j\in\mathcal{D}} \Delta u_j(Y^{\max}, y') \geq \sum_{j\in\mathcal{D}} \sum_{y=y'+1}^{Y^{\max}} \sigma_j^y$, which is a contradiction.

In conclusion, no coalition can jointly deviate from $\sigma_{\mathcal{N}}$ in a way that improves the sum of payoffs of its members. ■

Appendix B: Proofs of the results in the main text

Proof of Theorem 1. Consider a unit-by-unit contribution game $\Gamma = [N, (S_i, V_i)_{i\in N}]$ in economy $\mathbf{e^1}$. We consider a case in which $\mathcal{N} = N$ and $\mathcal{V}_i = V_i$ for all $i \in N$ and apply Lemmas A1, A3, A4, and A5 to Γ. Claims 1 and 2 are basic properties of economy $\mathbf{e^1}$.

Claim 1 In economy $\mathbf{e^1}$, $Y^{\max} = y^*$.

Proof of Claim 1. By the definition of $Y^{\max}$, $y^* \leq Y^{\max}$. Suppose that $y^* < Y^{\max}$. Since $\{y^*\} = \arg\max_{y\in\mathcal{Y}} \sum_{j\in N} u_j(y) - c(y)$, then $\sum_{j\in N} \Delta u_j(Y^{\max}, y^*) < \Delta c(Y^{\max}, y^*)$. Then, there exists $D \subsetneq N$

such that $Y(D) = Y^{\max}$ and $\sum_{j\in D} \Delta u_j(Y^{\max}, y^*) \geq \Delta c(Y^{\max}, y^*)$. In conclusion,

$$\sum_{j\in D} \Delta u_j(Y^{\max}, y^*) \geq \Delta c(Y^{\max}, y^*) > \sum_{j\in N} \Delta u_j(Y^{\max}, y^*).$$

However, it is impossible that $\sum_{j\in D} \Delta u_j(Y^{\max}, y^*) > \sum_{j\in N} \Delta u_j(Y^{\max}, y^*)$ because $D \subsetneq N$ and $\Delta u_j(y, y-1) \geq 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y \geq 1$. Hence, $Y^{\max} = y^*$ in $\mathbf{e^1}$. ||

Claim 2 In economy $\mathbf{e^1}$, (20) in Lemma A4 holds.

Proof of Claim 2. By Claim 1, $Y^{\max} = y^*$. Since y^* is the unique maximizer of $\sum_{j\in N} u_j(y) - c(y)$, then $\sum_{j\in N} \Delta u_j(y, y^*) < \Delta c(y, y^*)$ for all $y \in \mathcal{Y}$ such that $y \geq y^* + 1$. For all such y, since $\Delta u_j(y, y^*) \geq 0$ for all $j \in N$, we have $\sum_{j\in E} \Delta u_i(y, y^*) \leq \sum_{j\in N} \Delta u_j(y, y^*)$ for all $E \subseteq N$. Thus, for all $E \subseteq N$ and all $y \in \mathcal{Y}$ such that $y \geq y^* + 1$,

$$\sum_{j\in E} \Delta u_i(y, y^*) \leq \sum_{j\in N} \Delta u_j(y, y^*) < \Delta c(y, y^*).$$

Hence, (20) in Lemma A4 holds. ||

Proof of Theorem 1(i). Suppose, to the contrary, that there exists a Nash equilibrium $\sigma \in \mathbb{R}_+^{n\bar{y}}$ such that $y(\sigma) > y^*$. By the efficiency of y^*, $\sum_{j\in N} \Delta u_j(y(\sigma), y^*) < \Delta c(y(\sigma), y^*)$. Since σ is a Nash equilibrium, we obtain $\Delta c(y(\sigma), y^*) = \sum_{j\in N} \sum_{y=y^*+1}^{y(\sigma)} \sigma_j^y$ by (14). By these two conditions, $\sum_{j\in N} \Delta u_j(y(\sigma), y^*) < \sum_{j\in N} \sum_{y=y^*+1}^{y(\sigma)} \sigma_j^y$. This condition implies that there must be $l \in N$ such that $0 \leq \Delta u_l(y(\sigma), y^*) < \sum_{y=y^*+1}^{y(\sigma)} \sigma_l^y$. Now, consider a deviation by agent l such that she makes the same contributions from the first to the y^*-th unit and she makes no contributions to the level over y^*. If we denote the level after such a deviation by y', then $y' \in \{y^*, \ldots, y(\sigma) - 1\}$, and agent l gains $\sum_{y=y^*+1}^{y(\sigma)} \sigma_l^y - \Delta u_l(y(\sigma), y') > 0$,[11] which contradicts the supposition that σ is a Nash equilibrium.

Proof of Theorem 1(ii). Firstly, we show that there is a Nash equilibrium at which the project is undertaken at the level y^*.

Claim 3 There exists $\sigma^* \in \mathbb{R}_+^{n\bar{y}}$ such that

$$\begin{aligned} &\sigma_j^{*y} = 0 \text{ for all } j \in N \text{ and all } y \in \mathcal{Y} \text{ such that } y > y^*, \\ &\sum_{j\in N} \sigma_j^{*y} = \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq y^*, \\ \text{and } \ \Delta u_j(y^*, \hat{y}) \geq &\sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \text{ for all } j \in N \text{ and all } y \in \mathcal{Y} \text{ such that } 0 \leq \hat{y} \leq y^* - 1. \end{aligned} \tag{26}$$

Proof of Claim 3. Obviously, we can set $\sigma_j^{*y} = 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y > y^*$.

We construct $(\sigma_j^{*y})_{j\in N}$ for all y such that $1 \leq y \leq y^*$ by induction. We start with $y = y^*$. By Lemma 1, we obtain $\sum_{j\in N} \Delta u_j(y^*, y^* - 1) > \Delta c(y^*, y^* - 1)$. Thus, there exists $(\sigma_j^{y^*})_{j\in N} \in \mathbb{R}_+^n$ such that

$$\sum_{j\in N} \sigma_j^{*y^*} = \Delta c(y^*, y^* - 1) \text{ and } \Delta u_j(y^*, y^* - 1) \geq \sigma_j^{*y^*} \text{ for all } j \in N.$$

Given this $(\sigma_j^{*y^*})_{j\in N}$, we next construct $(\sigma_j^{*y^*-1})_{j\in N}$.

[11] Note that $\Delta u_l(y(\sigma), y') \leq \Delta u_l(y(\sigma), y^*) < \sum_{y=y^*+1}^{y(\sigma)} \sigma_l^y$.

Let $\hat{y} \in \mathcal{Y}$ be such that $1 \leq \hat{y} \leq y^* - 1$. Suppose that $(\sigma_j^{*y})_{j \in N}$ has been constructed for all $y \in \mathcal{Y}$ such that $\hat{y}+1 \leq y \leq y^*$. We now construct $(\sigma_j^{*\hat{y}})_{j \in N}$. By Lemma 1, $\sum_{j \in N} \Delta u_j(y^*, \hat{y}-1) > \Delta c(y^*, \hat{y}-1)$. This condition is equivalent to

$$\sum_{j \in N} \Delta u_j(y^*, \hat{y}) + \sum_{j \in N} \Delta u_j(\hat{y}, \hat{y}-1) > \Delta c(y^*, \hat{y}) + \Delta c(\hat{y}, \hat{y}-1)$$
$$= \sum_{y=\hat{y}+1}^{y^*} \sum_{j \in N} \sigma_j^{*y} + \Delta c(\hat{y}, \hat{y}-1).$$

Thus,

$$\sum_{j \in N} \left[\Delta u_j(y^*, \hat{y}) - \sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \right] + \sum_{j \in N} \Delta u_j(\hat{y}, \hat{y}-1) > \Delta c(\hat{y}, \hat{y}-1)$$

By the induction hypothesis, $\Delta u_j(y^*, \hat{y}) - \sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \geq 0$ for all $j \in N$. Hence, there exists $(\sigma_j^{*\hat{y}})_{j \in N}$ such that

$$\sum_{j \in N} \sigma_j^{*\hat{y}} = \Delta c(\hat{y}, \hat{y}-1) \text{ and } \Delta u_j(y^*, \hat{y}-1) - \sum_{y=\hat{y}+1}^{y^*} \sigma_j^{*y} \geq \sigma_j^{*\hat{y}} \text{ for all } j \in N.$$

||

At σ^*, the project is undertaken at the level y^*. Note that σ^* satisfies all conditions in Lemma A3; hence, it is a Nash equilibrium.

Claim 4 Every Nash equilibrium at which the project is undertaken at y^* is a strong Nash equilibrium with transfers.

Proof of Claim 4. Firstly, note that (22) in Lemma A5 holds since $\Delta u_j(y, y-1) \geq 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y \geq 1$. Secondly, by Claim 2, (20) in Lemma A4 holds. Thus, by Lemma A5, every Nash equilibrium at which the project is undertaken at the level y^* is a strong Nash equilibrium with transfers. ||

By Claim 4 and Remark 1, with and without transfers, every Nash equilibrium at which the project is undertaken at y^* is a strong Nash equilibrium and a coalition-proof Nash equilibrium. Note that by the definitions of strong Nash equilibria with and without transfers, all strong Nash equilibria with and without transfers must be Nash equilibria at which the project is undertaken at the level y^*. Hence, the sets of these two strong Nash equilibria coincide with the set of Nash equilibria with efficient implementation of the project.

Claim 5 All coalition-proof Nash equilibria with and without transfers are strong Nash equilibria with transfers.

Proof of Claim 5. Firstly, we show that every coalition-proof Nash equilibrium (without transfers) is a strong Nash equilibrium with transfers. Since the sets of strong Nash equilibria with and without transfers coincide, it is enough to show that without transfers every coalition-proof Nash equilibrium is a strong Nash equilibrium. Suppose, to the contrary, that there exists a coalition-proof Nash equilibrium $\sigma \in \mathbb{R}_+^{n\bar{y}}$ that is not a strong Nash equilibrium. Since the set of strong Nash equilibria coincides with that of Nash equilibria at which y^* is the level of the project, $y(\sigma)$ must be an inefficient level of the project. Hence, $y(\sigma) < y^*$ and $\sum_{j \in N} \Delta u_j(y^*, y(\sigma)) > \Delta c(y^*, y(\sigma))$. Similar to the construction of σ^* in (26), we construct $(\sigma_j'^y)_{j \in N}$ for all $y \in \mathcal{Y}$ such that $y(\sigma) + 1 \leq y \leq \bar{y}$ as follows:

- $\sigma_j'^y = 0$ for all $j \in N$ and all $y \in \mathcal{Y}$ such that $y > y^*$.
- $\sum_{j \in N} \sigma_j'^y = \Delta c(y, y-1)$ for all $y \in \mathcal{Y}$ such that $y(\sigma) + 1 \leq y \leq y^*$.
- $\Delta u_j(y^*, \hat{y}) \geq \sum_{y=\hat{y}+1}^{y^*} \sigma_j'^y$ for all $j \in N$ and all $\hat{y} \in \mathcal{Y}$ such that $y(\sigma) \leq \hat{y} \leq y^* - 1$.
- $\Delta u_j(y^*, y(\sigma)) > \sum_{y=y(\sigma)+1}^{y^*} \sigma_j'^y$ for all $j \in N$.

The last condition follows from $\sum_{j \in N} \Delta u_j(y^*, y(\sigma)) > \Delta c(y^*, y(\sigma))$. Combining $(\sigma_j'^y)_{y=y(\sigma)+1}^{\bar{y}}$ with $(\sigma_j^y)_{y=1}^{y(\sigma)}$ for all $j \in N$, we make a new strategy profile $\tilde{\sigma} \equiv ((\sigma_j^y)_{y=1}^{y(\sigma)}, (\sigma_j'^y)_{y=y(\sigma)+1}^{\bar{y}})_{j \in N}$. At $\tilde{\sigma}$, the project is undertaken at the level y^*, and by Lemma A3 it is a Nash equilibrium. By Lemma A5, $\tilde{\sigma}$ is a strong Nash equilibrium with transfers, and hence $\tilde{\sigma}$ is also coalition-proof without transfers. Since $\Delta u_j(y^*, y(\sigma)) > \sum_{y=y(\sigma)+1}^{y^*} \sigma_j'^y$ for all $j \in N$, $\tilde{\sigma}$ Pareto dominates σ. By Definition 2, σ cannot be a coalition-proof Nash equilibrium, which is a contradiction (see Remark 1(iii)).

Finally, note that we can show similarly that every coalition-proof Nash equilibrium with transfers is a strong Nash equilibrium with transfers. ||

In conclusion, we obtain that the five equilibrium sets coincide. ■

Proof of Proposition 1. ($\Leftarrow$) We show that $\sigma^* \in \mathbb{R}_+^{n\bar{y}}$ constructed in (8) is a Nash equilibrium. Firstly, note that by (1), (5), and (9),

$$\Delta u_j(y+1, y) \leq \Delta c(y+1, y) \text{ for all } j \in N \text{ and all } y \in \mathcal{Y} \text{ such that } y \geq y^*. \tag{27}$$

Considering the game Γ^0 with $\mathcal{N} = N^{y^*}$, we apply Lemma A3 to the game $\Gamma|\sigma^*_{N \setminus N^{y^*}}$. Clearly, we have $\Delta u_j(y, y-1) \geq 0$ for all $j \in N^{y^*}$ and all $y \in \mathcal{Y}$ such that $1 \leq y \leq y^*$; hence, (17) holds at $\sigma_{\mathcal{N}} = \sigma^*_{N^{y^*}}$. By (27), (18) holds at $\sigma_{\mathcal{N}} = \sigma^*_{N^{y^*}}$. (14)–(16) hold by the construction of $\sigma^*_{N^{y^*}}$. Thus, by Lemma A3, $\sigma^*_{N^{y^*}}$ is a Nash equilibrium of $\Gamma|\sigma^*_{N \setminus N^{y^*}}$.

By (27), no agent outside N^{y^*} is made better off if he unilaterally increases his contribution in such a way that the project is undertaken over the level y^*. Also, no agent outside N^{y^*} is made better off if he increases contributions to a level under y^* because the contributions from agents in N^{y^*} already cover the cost of y^* units. In conclusion, σ^* is a Nash equilibrium of the unit-by-unit contribution game.

($\Rightarrow$) Suppose, to the contrary, that there exists a Nash equilibrium σ such that $y(\sigma) = y^*$ and that $\Delta u_j(y^*+1, y^*) > \Delta c(y^*+1, y^*)$ for some $j \in N$. By applying Lemma A1, we have $\sigma_i^{y^*+1} = 0$ for all $i \in N$. Clearly, if agent j switches from $\sigma_j^{y^*+1} = 0$ to $\tilde{\sigma}_j^{y^*+1} = \Delta c(y^*+1, y^*)$, then his payoff increases by $\Delta u_j(y^*+1, y^*) - \Delta c(y^*+1, y^*) > 0$, which is a contradiction. ■

Proof of Proposition 2. ($\Rightarrow$) Suppose that there exists a Nash equilibrium σ such that $y(\sigma) > y^*$ but $Y^{\max} \leq y^*$. Since $Y^{\max} < y(\sigma)$, then $Y(D) < y(\sigma)$ for all $D \subseteq N$. Note especially that $Y(N^{y(\sigma)}) < y(\sigma)$.

Since σ is a Nash equilibrium, then $\sigma_j^{y(\sigma)} = 0$ for all $j \in N \setminus N^{y(\sigma)}$: if $\sigma_j^{y(\sigma)} > 0$ for some $j \in N \setminus N^{y(\sigma)}$, agent j is made better off by deviating from σ in a way that changes her contribution to the

$y(\sigma)$-th unit to zero and takes the same contribution to the other units as σ_j.[12] Thus,

$$\Delta c(y(\sigma), y(\sigma) - 1) = \sum_{j \in N^{y(\sigma)}} \sigma_j^{y(\sigma)}.$$

By the properties of the Nash equilibria in Lemma A2, we obtain $\Delta u_j(y(\sigma), y(\sigma) - 1) \geq \sigma_j^{y(\sigma)}$ for all $j \in N^{y(\sigma)}$, implying

$$\sum_{j \in N^{y(\sigma)}} \Delta u_j(y(\sigma), y(\sigma) - 1) \geq \Delta c(y(\sigma), y(\sigma) - 1).$$

By the weak concavity of u_j and the weak convexity of c, for all $y \in \mathcal{Y}$ such that $y \leq y(\sigma)$,

$$\sum_{j \in N^{y(\sigma)}} \Delta u_j(y, y - 1) \geq \sum_{j \in N^{y(\sigma)}} \Delta u_j(y(\sigma), y(\sigma) - 1) \geq \Delta c(y(\sigma), y(\sigma) - 1) \geq \Delta c(y, y - 1).$$

These inequalities imply that $Y(N^{y(\sigma)}) < y(\sigma)$ never holds; by these inequalities, if $Y(N^{y(\sigma)}) < y(\sigma)$, then $Y(N^{y(\sigma)})$ cannot be a unique maximizer of $\sum_{j \in N^{y(\sigma)}} u_j(y) - c(y)$.

($\Leftarrow$) Let $D \subsetneq N$ be such that $Y(D) = Y^{\max}$. By the definition of $Y(D)$, $\sum_{j \in D} \Delta u_j(Y^{\max}, Y^{\max} - 1) > c(Y^{\max}, Y^{\max} - 1)$. Since $\Delta u_j(Y^{\max}, Y^{\max} - 1) < 0$ for all $j \in N \backslash N^{Y^{\max}}$, then $\sum_{j \in D \cap N^{Y^{\max}}} \Delta u_j(Y^{\max}, Y^{\max} - 1) > c(Y^{\max}, Y^{\max} - 1)$. By this inequality, the weak concavity of u_j, and the weak convexity of c,

$$\sum_{j \in D \cap N^{Y^{\max}}} \Delta u_j(y, y - 1) > \Delta c(y, y - 1) \quad \text{for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}.$$

By this condition, we can construct $\tilde{\sigma} \in \mathbb{R}_+^{n\bar{y}}$ such that

- If $Y^{\max} + 1 \leq y \leq \bar{y}$, then $\tilde{\sigma}_j^y = 0$ for all $j \in N$.
- If $1 \leq y \leq Y^{\max}$, then $\tilde{\sigma}_j^y = 0$ for all $j \in N \backslash (D \cap N^{Y^{\max}})$, $0 \leq \tilde{\sigma}_j^y \leq \Delta u_j(y, y - 1)$ for all $j \in D \cap N^{Y^{\max}}$, and $\sum_{j \in D \cap N^{Y^{\max}}} \tilde{\sigma}_j^y = \Delta c(y, y - 1)$.

Similar to the method in the proof of Proposition 1, if we apply Lemma A3 to $\Gamma | \tilde{\sigma}_{N \backslash (D \cap N^{Y^{\max}})}$, we can show that $\tilde{\sigma}$ is a Nash equilibrium of Γ.[13] ■

Proof of Lemma 5. Let $M \subsetneq N$ be such that $Y(M) = Y^{\max}$. Since $\{Y^{\max}\} = \arg\max_{y \in \mathcal{Y}} \sum_{j \in M} u_j(y) - c(y)$,

$$\sum_{j \in M} \Delta u_j(Y^{\max}, Y^{\max} - 1) > \Delta c(Y^{\max}, Y^{\max} - 1). \tag{28}$$

Excluding agents $j \in M$, if any, such that $\Delta u_j(Y^{\max}, Y^{\max} - 1) \leq 0$, we make $M^+ \equiv \{j \in M | \Delta u_i(Y^{\max}, Y^{\max} - 1) > 0\}$. Then, by (28), we obtain

$$\Delta c(Y^{\max}, Y^{\max} - 1) < \sum_{j \in M} \Delta u_j(Y^{\max}, Y^{\max} - 1) \leq \sum_{j \in M^+} \Delta u_j(Y^{\max}, Y^{\max} - 1).$$

[12] By this deviation, the level of the project decreases to $y(\sigma) - 1$ and j's payoff increases by $-\Delta u_j(y(\sigma), y(\sigma) - 1) + \sigma_j^{y(\sigma)} > 0$.

[13] Note that $\Delta u_j(Y^{\max} + 1, Y^{\max}) \leq \Delta c(Y^{\max} + 1, Y^{\max})$ for all $j \in N$ since $Y^{\max} = \max_{D' \subseteq N} Y(D')$.

By the weak concavity of u_j for all $j \in N$ and the weakly convexity of c,

$$\sum_{j \in M^+} \Delta u_j(y, y-1) > \Delta c(y, y-1) \text{ for all } y \in \mathcal{Y} \text{ such that } 1 \leq y \leq Y^{\max}. \tag{29}$$

By the weak concavity of u_j for all $j \in N$, if $\Delta u_j(Y^{\max}, Y^{\max}-1) > 0$, then $\Delta u_j(y, y-1) > 0$ for all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y^{\max}$. Hence, M^+ is a set that satisfies (10) and (11). ■

Proof of Proposition 3. We show that σ, constructed in (12), is coalition-proof with and without transfers. Suppose that a coalition $D \subsetneq N$ deviates from σ_D to $\sigma'_D \in \mathbb{R}_+^{|D|\bar{y}}$. Let $y' \equiv y(\sigma'_D, \sigma_{N \setminus D})$: y' is the level of the public project attained by this deviation. Trivially, note that the deviation is never profitable if $y' = Y^{\max}$.

Claim 6 Suppose that $y' \geq Y^{\max} + 1$. Then, $\sum_{j \in D} V_j(\sigma) \geq \sum_{j \in D} V_j(\sigma'_D, \sigma_{N \setminus D})$ and there is $i \in D$ such that $V_i(\sigma) \geq V_i(\sigma'_D, \sigma_{N \setminus D})$.

Proof of Claim 6. We prove this claim by Lemma A4. We consider the case of $\Gamma^0 = \Gamma$. Let $E \subseteq N$. Since $Y(E)$ is the unique maximizer of $\sum_{j \in E} u_j(y) - c(y)$,

$$\sum_{j \in E} \Delta u_j(Y(E)+1, Y(E)) < \Delta c(Y(E)+1, Y(E)).$$

Thus, by the weak concavity of u_j and the weak convexity of c, for all $y \in \mathcal{Y}$ such that $y \geq Y(E)$,

$$\sum_{j \in E} \Delta u_j(y+1, y) \leq \sum_{j \in E} \Delta u_j(Y(E)+1, Y(E)) < c(Y(E)+1, Y(E)) \leq c(y+1, y). \tag{30}$$

Note that by the definition of $Y^{\max}$, $Y(E) \leq Y^{\max}$. By this condition and (30),

$$\sum_{j \in E} \Delta u_j(y, Y^{\max}) < c(y, Y^{\max}) \tag{31}$$

for all $y \in \mathcal{Y}$ such that $y \geq Y^{\max}+1$. Thus, in Γ, (20) holds. By Lemma A4, $\sum_{j \in D} V_j(\sigma) \geq \sum_{j \in D} V_j(\sigma'_D, \sigma_{N \setminus D})$, implying that there exists $i \in D$ such that $V_i(\sigma) \geq V_i(\sigma'_D, \sigma_{N \setminus D})$. ||

Firstly, we show that σ is a coalition-proof Nash equilibrium (without transfers). Suppose, to the contrary, that σ is not coalition-proof. Then, there exist a coalition $D \subseteq N$ and $\sigma'_D \in \mathbb{R}_+^{|D|\bar{y}}$ such that $V_i(\sigma) < V_i(\sigma'_D, \sigma_{N \setminus D})$ for all $i \in D$, and σ'_D is a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N \setminus D}$. By Claim 6, if $y' \geq Y^{\max} + 1$, then it is impossible for the deviation by D to be profitable, irrespective of the self-enforcing property of σ'_D. Hence, we need to consider the case of $y' \leq Y^{\max} - 1$.

Claim 7 Suppose that $y' \leq Y^{\max} - 1$. In the restricted game $\Gamma|(\sigma_{N \setminus D}, \sigma'_{D \setminus \mathcal{M}})$,

(7. i) $((\sigma_i'^{y})_{y=1}^{y'}, (\sigma_i^{y})_{y=y'+1}^{\bar{y}})_{i \in D \cap \mathcal{M}}$, where $(\sigma_i^{y})_{y=y'+1}^{\bar{y}}$ is defined in (12) for all $i \in D \cap \mathcal{M}$, is a Nash equilibrium at which the project is undertaken at the level $Y^{\max}$,

(7. ii) every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers, and

(7. iii) $\sigma'_{D\cap\mathcal{M}}$ is strictly Pareto dominated by $((\sigma_i'^y)_{y=1}^{y'}, (\sigma_i^y)_{y=y'+1}^{\bar{y}})_{i\in D\cap\mathcal{M}}$.

Proof of Claim 7. For notational simplicity, denote $\sigma'_{N\setminus(D\cap\mathcal{M})} \equiv (\sigma_{N\setminus D}, \sigma'_{D\setminus\mathcal{M}})$ and $\sigma^{**}_{D\cap\mathcal{M}} \equiv ((\sigma_i'^y)_{y=1}^{y'}, (\sigma_i^y)_{y=y'+1}^{\bar{y}})_{i\in D\cap\mathcal{M}}$.

If the level of the public project declines to y' by this deviation, some agents in $\mathcal{M}$ join in this deviation; otherwise, the level of the project never decreases (note that at σ, no agent outside $\mathcal{M}$ contributes). Hence, $D \cap \mathcal{M} \neq \emptyset$.

Proof of (7.i) We apply Lemma A3 to show that $\sigma^{**}_{D\cap\mathcal{M}}$ is a Nash equilibrium, considering $\Gamma^0 = \Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$; that is, $\mathcal{N}$ in Lemma A3 is equal to $D\cap\mathcal{M}$. Obviously, at $\sigma^{**}_{D\cap\mathcal{M}}$, the project is undertaken at $Y^{\max}$ in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$. By (10), (17) holds at $\sigma_{\mathcal{N}} = \sigma^{**}_{D\cap\mathcal{M}}$ in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$. (31) implies that (18) holds at $\sigma_{\mathcal{N}} = \sigma^{**}_{D\cap\mathcal{M}}$ in this game. By the fact that σ'_D must be a Nash equilibrium in $\Gamma|\sigma_{N\setminus D}$, Lemmas A1 and A2, and the construction of σ in (12), $\sigma^{**}_{D\cap\mathcal{M}}$ satisfies (14)–(16) in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$. Hence, by Lemma A3, $\sigma^{**}_{D\cap\mathcal{M}}$ is a Nash equilibrium.

Proof of (7.ii) We show this by Lemma A5. We consider Γ^0 in which $\mathcal{N} = \mathcal{D} \cap \mathcal{M}$ and $\mathcal{V}_j(\bullet) = V_j(\bullet, \sigma'_{N\setminus(D\cap\mathcal{M})})$ for all $j \in D\cap\mathcal{M}$. By (10) of Lemma 5, we obtain $\Delta u_j(y, y-1) > 0$ for all $j \in D\cap\mathcal{M}$ and all $y \in \mathcal{Y}$ such that $1 \leq y \leq Y^{\max}$. Hence, (22) holds.

We now prove that (20) holds. Let $E \subseteq D\cap\mathcal{M}$. In a way similar to (30),

$$\sum_{j\in E} \Delta u_j(y+1, y) \leq \sum_{j\in E} \Delta u_j(Y(E)+1, Y(E)) < c(Y(E)+1, Y(E)) \leq c(y+1, y).$$

for all $y \in \mathcal{Y}$ such that $y \geq Y(E)$. In a similar way to (31),

$$\sum_{j\in E} \Delta u_j(y, Y^{\max}) < c(y, Y^{\max}).$$

for all $y \in \mathcal{Y}$ such that $y \geq Y^{\max}+1$. Thus, (20) holds. By Lemma A5, every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$.

Proof of (7.iii) For all $i \in D\cap\mathcal{M}$, i's payoff at $\sigma'_{D\cap\mathcal{M}}$ is $u_i(y') - \sum_{y=1}^{y'} \sigma_i'^y$, while his payoff at $\sigma^{**}_{D\cap\mathcal{M}}$ is

$$u_i(Y^{\max}) - \sum_{y=1}^{y'} \sigma_i'^y - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y = u_i(y') - \sum_{y=1}^{y'} \sigma_i'^y + \Delta u_i(Y^{\max}, y') - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y.$$

By (13), $\Delta u_i(Y^{\max}, y') - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y > 0$ for all $i \in D\cap\mathcal{M}$; hence,

$$u_i(y') - \sum_{y=1}^{y'} \sigma_i'^y < u_i(Y^{\max}) - \sum_{y=1}^{y'} \sigma_i'^y - \sum_{y=y'+1}^{Y^{\max}} \sigma_i^y \tag{32}$$

for all $i \in D\cap\mathcal{M}$. ||

By (7.i) and (7.ii) of Claim 7, $\sigma^{**}_{D\cap\mathcal{M}}$ is a strong Nash equilibrium with transfers of $\Gamma|(\sigma_{N\setminus D}, \sigma'_{D\setminus\mathcal{M}})$; hence, it is a coalition-proof Nash equilibrium of the restricted game. Note that by the definition of coalition-proof Nash equilibria, no coalition-proof Nash equilibrium is Pareto-dominated by other coalition-proof Nash equilibria (see Remark 1(iii)). Thus, by (7.iii) of this claim, $\sigma'_{D\cap\mathcal{M}}$ is not coalition-proof in $\Gamma|\sigma'_{N\setminus(D\cap\mathcal{M})}$, which in turn implies that σ'_D is not a coalition-proof Nash equilibrium of

$\Gamma|\sigma_{N\setminus D}$.[14] This is a contradiction. Thus, by Claims 6 and 7, σ is a coalition-proof Nash equilibrium of Γ.

Secondly, we can similarly show that σ is a coalition-proof Nash equilibrium with transfers. We can prove that if σ'_D is a coalition-proof Nash equilibrium with transfers of $\Gamma|\sigma_{N\setminus D}$, then in $\Gamma|(\sigma_{N\setminus D}, \sigma'_{D\setminus\mathcal{M}})$, $\sigma^{**}_{D\cap\mathcal{M}}$ is a strong Nash equilibrium with transfers at which the project is undertaken at the level $Y^{\max}$ in a similar way to (7.ii) of Claim 7. Moreover,

$$\sum_{j\in D\cap\mathcal{M}}\left(u_j(y') - \sum_{y=1}^{y'}\sigma'^{y}_j\right) < \sum_{j\in D\cap\mathcal{M}}\left(u_j(Y^{\max}) - \sum_{y=1}^{y'}\sigma'^{y}_j - \sum_{y=y'+1}^{Y^{\max}}\sigma^{y}_j\right),$$

which is obtained by summing (32) over $j\in D\cap\mathcal{M}$. By applying the reasoning in Remark 1(iv), σ'_D is not coalition-proof with transfers in $\Gamma|\sigma_{N\setminus D}$. Hence, σ is also coalition-proof with transfers. ■

Proof of Proposition 4. Firstly, note that by Lemma A1, since $\hat{\sigma}$ is a Nash equilibrium, $\sum_{j\in N}\hat{\sigma}^y_j = \Delta c(y, y-1)$ for all $y\in\mathcal{Y}$ such that $1\le y\le y(\hat{\sigma})$ and $\sum_{j\in N}\hat{\sigma}^y_j = 0$ for all $y\in\mathcal{Y}$ such that $y(\hat{\sigma})+1\le y\le Y^{\max}$. Denote $\sigma^{**}_{\mathcal{M}}\equiv((\hat{\sigma}^y_i)^{y(\hat{\sigma})}_{y=1}, (\sigma^y_i)^{\bar{y}}_{y=y(\hat{\sigma})+1})_{i\in\mathcal{M}}$, in which $((\sigma^y_i)^{\bar{y}}_{y=y(\hat{\sigma})+1})_{i\in\mathcal{M}}$ is defined in (12). As in (7.i) of Claim 7, $\sigma^{**}_{\mathcal{M}}$ is shown to be a Nash equilibrium at which the project is undertaken at $Y^{\max}$ in $\Gamma|\hat{\sigma}_{N\setminus\mathcal{M}}$.

Claim 8 In $\Gamma|\hat{\sigma}_{N\setminus\mathcal{M}}$, (8.i) $V_j(\sigma^{**}_{\mathcal{M}}, \hat{\sigma}_{N\setminus\mathcal{M}}) > V_j(\hat{\sigma})$ for all $j\in\mathcal{M}$ and (8.ii) $\sigma^{**}_{\mathcal{M}}$ is a strong Nash equilibrium with transfers.

Proof of Claim 8. The proof of (8.i) is similar to that of (7.iii) of Claim 7. Note that $\Delta u_j(Y^{\max}, y(\hat{\sigma})) > \sum^{Y^{\max}}_{y=y(\hat{\sigma})+1}\sigma^y_j$ for all $j\in\mathcal{M}$ (see (13)).

(8.ii) is shown by Lemma A5. We consider Γ^0 such that $\mathcal{N}=\mathcal{M}$ and $\mathcal{V}_j(\bullet) = V_j(\bullet, \hat{\sigma}_{N\setminus\mathcal{M}})$ for all $j\in\mathcal{M}$. By (10) of Lemma 5, we have $\Delta u_j(y, y-1) > 0$ for all $j\in\mathcal{M}$ and all $y\in\mathcal{Y}$ such that $1\le y\le Y^{\max}$. Hence, (22) holds.

We now prove that (20) holds. Let $E\subseteq\mathcal{M}$. Since $Y(E)$ uniquely maximizes $\sum_{j\in E}u_j(y) - c(y)$,

$$\sum_{j\in E}\Delta u_j(Y(E)+1, Y(E)) < \Delta c(Y(E)+1, Y(E)).$$

By (1) and (5), for all $y\in\mathcal{Y}$ such that $y\ge Y(E)$,

$$\sum_{j\in E}\Delta u_j(y+1, y) \le \sum_{j\in E}\Delta u_j(Y(E)+1, Y(E)) < c(Y(E)+1, Y(E)) \le c(y+1, y).$$

By this condition and $Y(E)\le Y^{\max}$, we obtain $\sum_{j\in E}\Delta u_j(y, Y^{\max}) < c(y, Y^{\max})$ for all $y\in\mathcal{Y}$ such that $y\ge Y^{\max}+1$. Thus, (20) holds.

By Lemma A5, every Nash equilibrium at which the project is undertaken at the level $Y^{\max}$ is a strong Nash equilibrium with transfers. Since $\sigma^{**}_{\mathcal{M}}$ is a Nash equilibrium at which the project is undertaken at $Y^{\max}$, it is also a strong Nash equilibrium with transfers. ||

[14] Note that if σ'_D is a coalition-proof Nash equilibrium of $\Gamma|\sigma_{N\setminus D}$, then σ'_E is also coalition-proof of the corresponding restricted game for all $E\subsetneq D$.

We can show by Claim 8 that $\hat{\sigma}$ is not coalition-proof with or without transfers in Γ, similarly with the last two paragraphs of the proof of Proposition 3. ■

Proof of Theorem 3. Suppose that $Y^{\max} = y^*$. Then, by Propositions 3 and 4, the level of the project supported at coalition-proof Nash equilibria with and without transfers is y^* or higher. However, by Theorem 2(i), there exists no Nash equilibrium at which the project is undertaken over y^*. Hence, y^* is a unique level of the project supported at coalition-proof Nash equilibria.

Suppose that $Y^{\max} > y^*$. By Proposition 3, there exists a coalition-proof Nash equilibrium at which the project is undertaken over y^*. By Proposition 4, even if there exist Nash equilibria that support a level that is less than or equal to y^*, they are never coalition-proof with or without transfers in Γ. Hence, the public project is excessively undertaken at the coalition-proof Nash equilibria. ■

Proof of Proposition 5. It is enough to provide an example of the economy. We reconsider the economy specified in Example 1. Remember that $\mathcal{Y} = \{0, 1, 2\}$, $c(y) = 10y$ for all $y \in \mathcal{Y}$, $N = \{1, 2\}$, $u_1(1) = 4$, $u_1(2) = 1$, $u_2(1) = 12$, $u_2(2) = 23$, and $y^* = 1$. We show that no $\sigma \in \prod_{j \in N} S_j$ such that $y(\sigma) = 1$ is a Nash equilibrium. Take a strategy profile σ with $y(\sigma) = 1$. Note that $\sum_{j \in N} \sigma_j^2 < 10$, since $y(\sigma) = 1$. We obtain $V_i(\sigma) = u_i(1) - \sum_{y=1}^{2} t_i^y(\sigma_i^y)$ for each $i \in N$.

Firstly, suppose that $\sigma_i^2 > 0$ for some $i \in N$. Then, the payoff to agent i at σ is $u_i(1) - t_1^1(\sigma_i^1) - \sigma_i^2$. If agent i switches from $\sigma_i^2 > 0$ to $\sigma_i^{2\prime} = 0$ keeping the contribution to the first unit the same, then she can still enjoy the public project at one unit by $\sum_{j \in N} \sigma_j^2 < 10$ and receives the payoff $u_i(1) - t_1^1(\sigma_i^1)$, which is greater than the payoff before the switch.

Secondly, suppose that $\sigma_j^2 = 0$ for each $j \in N$. Then, if agent 2 keeps σ_2^1 the same and changes a contribution to the second unit from σ_2^2 to $\tilde{\sigma}_2^2 = \Delta c(2, 1)$, then the second unit is provided and he is made better off (note that $\Delta u_2(2, 1) > \Delta c(2, 1)$).

Finally, suppose that $\sigma_j^2 \leq 0$ for each $j \in N$ and $\sigma_i^2 < 0$ for at least one $i \in N$. Suppose that agent i switches from $\sigma_i^2 < 0$ to $\sigma_i^{2\prime} = 0$ keeping the contribution to the first unit the same. Then, after the switch, the project level is one unit since $\sigma_i^2 + \sigma_k^2 < \sigma_k^2 \leq 0 < 10$, where $k \neq i$. Thus, after the switch, agent 1 can still enjoy one unit of the project and receives the payoff $u_i(1) - t_1^1(\sigma_i^1)$, which is greater than $u_i(1) - t_1^1(\sigma_i^1) - t_1^2(\sigma_i^2)$.

In conclusion, no $\sigma \in \prod_{j \in N} S_j$ such that $y(\sigma) = 1$ is a Nash equilibrium. ■

Appendix C: Example of coalition-proofness

Consider the three-player game in Table 1.[15] We assume that the payoffs in this table are transferable among members of a coalition, if one forms. There are two pure-strategy Nash equilibria: (x_2, y_2, z_1) and (x_1, y_1, z_2). The former is a coalition-proof Nash equilibrium, but not one with transfers. The latter is a coalition-proof Nash equilibrium with transfers, but not a coalition-proof Nash equilibrium. Thus, the two sets of coalition-proof Nash equilibria are nonempty and disjoint.

[15]In this matrix, agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each cell is agent 1's payoff, the second is agent 2's, and the third is agent 3's.

Table 1: Appendix C

	y_1	y_2
x_1	4.5, 2, 0	1, 1, 1
x_2	1, 1, 1	3, 3, 5

z_1

	y_1	y_2
x_1	2, 2, 3	1, 1, 3
x_2	1, 1, 3	0, 0, 3

z_2

Acknowledgments. I thank Midori Hirokawa, Ryoji Sawa, Kaname Miyagishima, and Masuyuki Nishijima for helpful comments and suggestions. I am also grateful to the participants in the Asia Meeting of the Economic Society 2016 and the seminars at Yokohama City University and Hosei University. I gratefully acknowledge financial support from JSPS KAKENHI Grant-in-Aid for Scientific Research JP15H03349 and JP15K03361.

References

[1] Aumann, R.J. (1959). Acceptable points in general cooperative n-person games. In Contributions to the theory of games IV, edited by A.W. Tucker and R.D. Luce, 287–324. Princeton: Princeton University Press.

[2] Bagnoli, M. and B.L. Lipman (1989). Provision of public goods: Fully implementing the core through private contributions. Review of Economic Studies 56, 583–601.

[3] Bernheim, D.B., B. Peleg, and M. Whinston (1987). Coalition-proof Nash equilibria I: Concepts. Journal of Economic Theory 42, 1–12.

[4] Brânzei, R., E. Inarra, S. Tijs, and J.M. Zarzuelo (2005). Cooperation by asymmetric agents in a joint project. Journal of Public Economic Theory 7, 623–640.

[5] Gradstein, M., and S. Nitzan (1990). Binary participation and incremental provision of public goods. Social Choice and Welfare 7, 171–192.

[6] Laussel, D., and M. Le Breton (2001). Conflict and cooperation: The structure of equilibrium payoffs in common agency. Journal of Economic Theory, 100, 93–128.

[7] Nishimura, Y., and R. Shinohara (2013). A voluntary participation game through a unit-by-unit cost share mechanism of a non-excludable public good. Social Choice and Welfare 40, 793–814.

[8] Saijo, T., and T. Yamato (1999). A voluntary participation game with a non-excludable public good. Journal of Economic Theory 84, 227–242.

[9] Sasaki, K., T. Ohashi, and A. Ando (1997). High-speed rail transit impact on regional system: Does the Shinkansen contribute to dispersion? Annals of Regional Science 31, 77–98.

[10] Silvestre, J. (2012) Public microeconomics: Efficiency and equity in public policy. Edward Elgar Publishing.

[11] Shinohara, R. (2009). The possibility of efficient provision of a public good in voluntary participation games. Social Choice and Welfare 32, 367–387.

[12] Shinohara, R. (2014). Participation and demand levels for a joint project. Social Choice and Welfare 43, 925–952.

[13] Ureña, J.M., P. Menerault, and M. Gramendia (2009). The high-speed rail challenge for big intermediate cities: A national, regional and local perspective. Cities 26, 266–279.

[14] Yu, Y. (2005). Public goods provision: Unit-by-unit cost-sharing rules and the core. Review of Economic Design 9, 363–376.

Instructions for Contributor

1. The text should be in single-column format and typed in English using double spacing and wide margins on paper of approximately 21cm by 29.6cm. Each manuscript should be organized in the following order: title page, abstract of 100 ～ 150 words, JEL classification number(s), text (with notes, references, and captions for figures and tables), acknowledgement, The title page consists of the title of the paper, the full name(s) and the institutional affiliation of the author(s).

2. Please make sure not to embed "graphically designed" figures or tables, but prepare these using the PC software's facility. Don't import tables or figures into the text file but, instead indicate their approximate locations directly in the text. Only monochrome figure can be accepted. Color figures or illustrations must be converted to their monochrome counterparts.

3. Original articles only will be considered. Copyright to the article is transferred to the Institute of Comparative Economic Studies, Hosei University, effective if and when the article is accepted for publication in the *Journal of International Economic Studies*. The length of an original article should not exceed 8000 words and thirty pages in draft, including an abstract, tables, figures, acknowledgements, notes and references.

4. Notes should be numbered as a following example:
   At this date, the level of urbanization[2] can be estimated at 12.3 per cent.

5. References should be cited in the text by giving the surname of the author(s) with the year of publication as following examples:
   This is due to the fact that the process of economic development and modernization began first in the less urbanized countries (Bairoch, 1985, Chap. 16);
   Muth (1969) shows that the income elasticity of demand for housing is greater than unity;
   Thus, 'the apparent trend to a large non-proletarian "middle-class" has resolved itself into the creation of a large proletariat in a new form' (Braverman, 1974, p. 355).
   The full list of references should be given alphabetically by the first author's family name, at the end of the manuscript. They should be written using the following examples.
   Akabane, Y. (1970), Keizai Togo to Kokumin Keizai (Economic Integration and National Economy), in Miyazaki, Y.*et al. Gendai Shihon Shugi-ron* (*Modern Capitalism*), Chikuma Shobo, Tokyo, pp. 173-232.
   Esho, H. (1981), Tojo-koku Hi-seido-teki Noson Shin'yo Shijoron (Towards the Analysis on the Non-Institutional Rural Credit Markets in Developing Countries), *Keizai Shirin* (The Hosei *University Economic Review*), Vol. XLIII, No. 4, pp.113-65.
   Gannicott, K. (1986), Women, Wages, and Discrimination : Some Evidence from Taiwan, *Economic Development and Cultural Change*, Vol. 34, pp.721-30.
   Gerschenkron, A.(1962), *Economic Backwardness in Historical Perspective: A Book of Essays,* Harvard University Press, Cambridge.
   Minami, R. (1981), *Nihon no Keizai Hatten* (*Economic Development of Japan*), Tokyo Keizai Shimpo-Sha, Tokyo.
   Weiskopf, T.E. (1971), Alternative Patterns of Import Substitution in India, in Chenery, H. B. (ed) *Studies in Development Planning*, Harvard University Press, Cambridge and London, pp.95-121.

6. Authors will receive the first proofs for correction. Major revisions of the text at that time will not be possible. Fifty copies of each printing will be provided to the author at the expense of the Institute.

7. Manuscripts should be e-mailed both in source files (which means in any of the document file and/or spreadsheet file) and PDF format to jies@ml.hosei.ac.jp, Editorial Board for JIES.
   The Institute of Comparative Economic Studies
   Hosei University
   4342 Aihara-machi, Machida-shi
   Tokyo 194-0298, Japan

8. Inquiries may be made to the Institute at the following e-mail address:
   e-mail: jies@ml.hosei.ac.jp

ICES